CHAPTER 19

FACTORIAL TREATMENT ARRANGEMENTS

19.1 Inadequacy of the single-factor approach

19.1.1 In carrying out a variety trial it is necessary to decide on suitable methods of cultivation, rates of sowing, types and rates of application of fertilizer, etc. These would probably be determined in the light of standard practices in the district (although these may or may not be the best) or on past experimental results with a particular variety, but it may so happen that a new variety may perform best under a different set of conditions. Similarly, a fertilizer trial with a particular crop must necessarily be carried out with some chosen variety, usually that in commonest use, but it may happen that the same treatments would perform differently with other varieties.

19.1.2 The concept of a **factor** applied at various quantitative **levels** has already been introduced (§§ 17.3.4 and 17.17.1). For example, if in a field experiment superphosphate were applied in three different quantities (0, 100, 200 lb. per acre), then superphosphate is said to be a factor with three levels of application. Levels may be qualitative as well as quantitative. Thus, different varieties constitute the levels of a factor "varieties"; superphosphate, rock phosphate, etc. may be the levels of a factor "type of phosphate", comparable applications being arranged to contain equivalent quantities of phosphoric oxide or phosphorus, and so on.

19.1.3 Suppose we wish to combine a variety trial (4 varieties) with a fertilizer trial (3 levels of phosphate: nil, 100 lb. per acre, 200 lb. per acre). Under the classical method of experimentation in which only one factor is allowed to vary at a time, we should put down the varietal trial three times, once with each of the three levels of phosphate, and we should also have to test the three levels of phosphate in separate experiments with each of the varieties in turn. An alternative method would be to vary both factors simultaneously in a single experiment, which we may do by letting our treatments consist of *all combinations of one level from each factor*, twelve in all. This type of **factorial or multiple factor experiment** not only enables us to answer all the questions which would be answered by the seven separate experiments above, and much more economically, but also provides additional information. For example, we might find that the phosphate exerted an effect on all four varieties, but that it did not affect all varieties equally. Or we might find that there were significant differences between the mean yields of the varieties over all levels of the fertilizer, but that these differences varied at individual levels. This information can be supplied only by the factorial experiment, since under the single-factor scheme such variation may be due to the fact that the different experiments are necessarily on different sites.

19.1.4 The concept of the factorial experiment is due to Fisher, and was first put forward in 1926. The method was employed at Rothamsted (Yates also making a considerable contribution to its development), but it was not until the publication of *The design of experiments* (1935) and a paper by Yates, *Complex experiments* (also 1935), that these ideas were brought to the attention of a wider audience. So well established was the principle of keeping all conditions constant except the one factor under test, that the general adoption of factorial experimentation constitutes a revolution in the history of experimentation second only to the adoption of the Fisherian principles of experimental design in conjunction with the method of analysis of variance. In the quotation from Planck in Chapter 2 an experiment is described as "a question which science poses to nature". According to Fisher it is "a wholly mistaken view" that "we must ask Nature few questions, or, ideally, one question, at a time". Rather will Nature "best respond to a logical and carefully thought out questionnaire; indeed, if we ask her a single question, she will often refuse to answer until some other topic has been discussed".

19.2 Notation

19.2.1 Although the representation of a factor by a single (capital) letter is common (cf. § 17.17.22), it is usual to denote **treatment combinations** by lower-case letters. For example, if 4 varieties (V) are represented as v_1, v_2, v_3, v_4 and 3 quantitative levels of phosphate (P) as p_0, p_1, p_2, then all the treatment combinations of a factorial experiment with these four varieties and these three levels of phsophate would be represented as

$$v_1p_0 \quad v_2p_0 \quad v_3p_0 \quad v_4p_0 \quad v_1p_1 \quad v_2p_1 \quad v_3p_1 \quad v_4p_1 \quad v_1p_2 \quad v_2p_2 \quad v_3p_2 \quad v_4p_2.$$

This arrangement is termed a "4 × 3 factorial experiment" (4 levels of V, 3 levels of P).

19.2.2 Obviously, the treatment symbolized as v_2p_1, for example, indicates Variety 2 with the *second* level of phosphate applied. The symbol 0 is very commonly used for the lowest level of a factor, probably because the lowest level is often a zero application, but the use sometimes extends even to a factor like "varieties".

19.2.3 Considerable variation in notation is encountered. An illustration of this is that v_1p_0, v_2p_0, v_3p_0, v_4p_0 could be written as v_1, v_2, v_3, v_4, seeing that no phosphate is applied.

19.3 Designs for factorial experiments

19.3.1 Factorial experiments are sometimes referred to as "factorial designs". The factorial arrangement, however, really comes under the heading of the "treatments design", and the design of a factorial experiment can always be one of the three basic designs (subject to limitations mentioned in §§ 13.1.2, 13.13.2, and 14.8.2). This point has already been made in §§ 11.5.4 and 13.6.11.

19.3.2 Limitations imposed by the large number of treatment combinations required for factorial experiments constitute a serious problem which will be discussed later. However, it is immediately apparent that only for relatively few factorial experiments will the Latin square be a convenient design. With more complex factorials, the problem becomes so acute that special designs have to be devised for efficient control of error. Perhaps for this reason the use of the term "factorial designs" has some justification.

19.4 Independence and interaction of factors

19.4.1 Two factors A and B are said to be **independent** if the effects of factor A (i.e. the responses to the levels of A) are the same at all levels of factor B. This can occur only if a reciprocal relationship holds, such that the effects of B are the same at all levels of A.

19.4.2 When two factors are not independent, the effects of factor A vary over the different levels of B, and *vice versa*. In other words, the effects of A *depend* on the particular level of B, and *vice versa*. The factors are then said to **interact** and we speak of the **interaction** between the factors A and B, "the interaction $A \times B$", or "the interaction AB". In § 19.1.1, if the fertilizer factor did perform differently with different varieties, there would be a "varieties $\times$ fertilizer interaction", i.e. an interaction between the factors "varieties" and "fertilizer". Similarly, the situation referred to in § 19.1.3, i.e. where the phosphate effects were not the same for all varieties or where the varietal differences varied at the different phosphate levels, would represent a varieties $\times$ phosphate interaction, symbolized as $V \times P$ or VP.

19.4.3 In order to investigate the independence or interaction of two factors it is necessary to draw up an **interaction table**, e.g. for the 4×3 varieties-phosphate example:

Table 19.1: Arrangement of treatment combinations into an interaction table

Varieties	Level of phosphate		
	p_0	p_1	p_2
v_1	v_1p_0	v_1p_1	v_1p_2
v_2	v_2p_0	v_2p_1	v_2p_2
v_3	v_3p_0	v_3p_1	v_3p_2
v_4	v_4p_0	v_4p_1	v_4p_2

This is simply a rectangular table with rows and columns corresponding to the levels of the factors concerned, it being immaterial which factor corresponds to rows and which to columns. Because all treatment combinations are represented in the experiment, all cells in the table are occupied. In practice yields are entered corresponding to each treatment combination in the appropriate cell.

19.4.4 Comprehension of what is meant by independence and interaction may be assisted by consideration of the simplest possible case—the 2×2 table. Consider varieties at 2 levels (v_1, v_2) and phosphate at 2 levels (p_0=nil, p_1). Suppose that the true mean yield of treatment v_1p_0 is a, that the true varietal difference in the absence of phosphate ($v_2p_0 - v_1p_0$) is b, and that the true response to phosphate with Variety 1 ($p_1v_1 - p_0v_1$) is c. This information allows us to enter the true yields for three of the cells of the interaction table:

	p_0	p_1
v_1	a	$a + c$
v_2	$a + b$	?

If the true yield of v_2p_1 is $a + b + c$, then the following statements are true:

(1) The effects of the factors are additive, i.e. the effect of the varietal difference (at level p_0) and the response to phosphate (with v_1) may be added to give the combined effect measured by $v_2p_1 - v_1p_0$.

(2) The response to P for v_1, viz. $p_1v_1 - p_0v_1 = (a + c) - a = c$, is the same as for v_2, viz. $p_1v_2 - p_0v_2 = (a + b + c) - (a + b) = c$, and similarly, the varietal difference at level p_0, viz. $v_2p_0 - v_1p_0 = (a + b) - a = b$, is the same as that at level p_1, viz. $v_2p_1 - v_1p_1 = (a + b + c) - (a + c) = b$.

In this event the factors are said to act independently in the sense indicated in § 19.4.2 and also in the sense that treatment effects associated with the levels of one factor are added to the corresponding effects of the second factor to give the joint effects associated with the various treatment combinations.

On the other hand, if the true yield of v_2p_1 is not $a + b + c$, the effects of the two factors are not simply additive, and they are said to interact. The following numerical examples with $a = 12$, $b = 3$, $c = 4$ will illustrate this:

Table 19.2: Numerical examples of independence and interaction in a 2×2 table

	p_0	p_1
v_1	12	16
v_2	15	19

(i) No interaction

	p_0	p_1
v_1	12	16
v_2	15	25

	p_0	p_1
v_1	12	16
v_2	15	13

(ii) and (iii) Interaction present

In Table 19.2 (ii) the response to phosphate is much more marked with v_2 than with v_1; equally the varietal difference is greater with phosphate applied than without. Table 19.2 (iii) illustrates the opposite state of affairs, where the response to phosphate actually becomes negative with v_2; in most instances of interaction the effects are less dramatic and reversals of responses are not normally anticipated.

19.4.5 In a general interaction table such as that in Table 19.1, the factors are independent only if the additive relationship holds for *every* 2×2 table obtained by selecting any two levels of each factor. Conversely, if the additive relationship fails to hold for *any* such 2×2 table, the factors interact. An alternative, but equivalent definition is that, if the pattern of effects corresponding to the levels of one factor is unaltered at all levels of the other factor, the factors are independent; otherwise they interact.

19.4.6 It must be carefully understood that the above discussion is in terms of *true* yields of treatment combinations. In practice the situation is complicated by the existence of experimental error, so that, even if there is in fact no interaction between the factors, there will always be an apparent interaction. It therefore becomes a question of being able to tell whether a table of observed yields with the appearance of Table 19.2 (ii) or (iii) does in fact represent a genuine treatment effect (interaction) or whether it is merely a question of a chance deviation from the state of affairs represented by Table 19.2 (i). A test of significance based on the null hypothesis of no interaction will permit a decision to be made whether such a deviation can be accounted for by experimental error or not.

19.4.7 Interaction has been defined as the failure of two factors to comply with a hypothesis of additive effects. It may be noted that other definitions of independent action (and hence of interaction) are possible, but the above is the simplest possible. If we can say that two factors are independent, this will permit a relatively simple statement of results, and the definition based on additive effects is one which will be approximately obeyed by many pairs of factors in practice.

19.4.8 Kempthorne has drawn attention to the fact that compliance or non-compliance with a hypothesis of additive effects may be due solely to the scale on which observations are measured. To illustrate this point consider the following 2 × 2 tables:

	a_0	a_1
b_0	11·6	22·9
b_1	18·3	36·3

	a_0	a_1
b_0	1·06	1·36
b_1	1·26	1·56

On the assumption that the above figures contain no experimental error, we would say that the table on the left exhibits a marked interaction AB, whereas that on the right exhibits independence. Yet to the degree of accuracy shown by the decimal places, the figures on the right are logarithms of those on the left. Hence if the latter represent yields, the factors interact on the yield scale, but not on the log (yield) scale.

19.4.9 In any case the interaction of two factors is a purely statistical concept. Hence, if two fertilizers "interact", this does not imply that any chemical reaction, for example, occurs between the fertilizers. What is actually responsible for an interaction may be a difficult thing to decide, but this is not a biometrical matter.

19.4.10 There is a resemblance between the idea of independence and interaction in an interaction table and independence and association in a contingency table (§ 15.15.2). The nature of a contingency table (containing frequencies) is, however, entirely different from that of an interaction table (containing variate-values) and the definitions of independence are very different.

19.5 Partitioning of Treatments S.S. in a two-factor experiment

19.5.1 It follows from § 19.3.1 that a partitioning of the Total S.S. can be made in accordance with whichever basic design is appropriate. The set of treatments in a two-factor arrangement is obviously classified, and we must proceed as indicated in § 11.5. The ordinary situation is that an orthogonal partitioning of the Treatments S.S. is possible.

19.5.2 Suppose that the means of the treatment combinations over all replications in the 4 × 3 varieties-phosphate example are denoted by y_{ij}, where i represents the i^{th} variety and j the j^{th} level of phosphate, and that we set up an interaction table of the treatment means thus:

Table 19.3: Symbolic interaction table for a 4 × 3 factorial arrangement

Variety	Level of P			Varietal means
	p_1	p_2	p_3	
v_1	y_{11}	y_{12}	y_{13}	y_{10}
v_2	y_{21}	y_{22}	y_{23}	y_{20}
v_3	y_{31}	y_{32}	y_{33}	y_{30}
v_4	y_{41}	y_{42}	y_{43}	y_{40}
Phosphate means	y_{01}	y_{02}	y_{03}	$\bar{y}$

Properly, the symbols for the treatment means would contain an additional zero suffix indicating means, but this additional zero is superfluous here. For obvious reasons it is impossible to have zero as a suffix representing the first level, and so the notation for levels of P has been changed accordingly to avoid possible confusion.

19.5.3 If the number of replications of each treatment combination is the same (= r), the Treatments S.S. is $r\sum_i\sum_j(y_{ij} - \bar{y})^2$. As noted in § 13.3.4, the randomized blocks partitioning ([13.2]) is true for any rectangular array, and may be applied here. The symbols r and t of [13.2] will, however, represent here the number of levels of P and V respectively, and both sides must be multiplied by the number of replications. The partitioning is:

$$r\sum_i\sum_j(y_{ij} - \bar{y})^2 = 3r\sum_i(y_{i0} - \bar{y})^2 + 4r\sum_j(y_{0j} - \bar{y})^2 + r\sum_i\sum_j(y_{ij} - y_{i0} - y_{0j} + \bar{y})^2.$$

More generally, if V and P have m and n levels respectively, the partitioning is

$$r\sum_i\sum_j(y_{ij} - \bar{y})^2 = nr\sum_i(y_{i0} - \bar{y})^2 + mr\sum_j(y_{0j} - \bar{y})^2 + r\sum_i\sum_j(y_{ij} - y_{i0} - y_{0j} + \bar{y})^2, \quad [19.1]$$

and the component S.S.'s on the R.H.S. have respectively $m - 1$, $n - 1$, and $(m - 1)(n - 1)$ D.F., adding to the Treatments D.F., viz. $mn - 1$.

19.6 Main effect of a factor

19.6.1 The first S.S. on the R.H.S. of [19.1] represents differences between the varietal means in the margin of Table 19.3. It is called the "Varieties S.S." or the S.S. for the **main effect** of varieties. There being m varieties, it has $m - 1$ D.F. (3 D.F. in the example).

19.6.2 The effect of the i^{th} variety, averaged over all levels of the second factor, is estimated by $y_{i0} - \bar{y}$. It may be termed the main effect, or average effect (over all levels of the second factor), of the i^{th} level of the factor "varieties". The S.S. of these deviations is therefore a S.S. of the main effects of varieties, but this term is most commonly used in the singular, probably because of the common use of factors at two levels, where there is only one such effect, the main effect having only 1 D.F.

19.6.3 Similarly, the second S.S. on the R.H.S. represents differences between the phosphate means in the margin of Table 19.3, and is termed the S.S. for the main effect of phosphate. There being n levels of phosphate, it has $n - 1$ D.F. (2 D.F. in the example).

19.6.4 Instead of referring to the S.S. for the main effect of varieties or phosphate, we often shorten this to the "S.S.(V)" or the "S.S.(P)", or even, where there is no possibility of confusion, to simply "V" or "P".

19.7 The Interaction S.S.

19.7.1 The remaining S.S. on the R.H.S. of [19.1] represents treatment effects after variation due to the main effects of the two factors has been eliminated.

19.7.2 We must now examine the nature of these residual treatment effects. Above we defined $y_{i0} - \bar{y}$ as the main effect of the i^{th} variety. Similarly, $y_{ij} - y_{0j}$ represents the effect of the i^{th} variety at the j^{th} level of phosphate, and the mean of $y_{ij} - y_{0j}$ over all values of j is $y_{i0} - \bar{y}$. Consequently the S.S. of the $y_{ij} - y_{0j}$ about their mean represents the variation in the effect of the i^{th} variety over the different levels of phosphate. Symbolically it is

$$\sum_j[(y_{ij} - y_{0j}) - (y_{i0} - \bar{y})]^2 = \sum_j(y_{ij} - y_{i0} - y_{0j} + \bar{y})^2,$$

and the multiplication by r comes in because y_{ij} is a mean over r replications. If we now sum the similar S.S.'s for all varieties, we obtain the final S.S. of [19.1], viz. $r\sum_i\sum_j(y_{ij} - y_{i0} - y_{0j} + \bar{y})^2$, which therefore represents the variation in the effects of the varieties over the different phosphate levels. From § 19.4.5 it is apparent that such variation represents interaction, and so this S.S. is called the "Interaction S.S.", the "S.S. $V \times P$", the "S.S.(VP)", or simply "VP". As with all Treatments S.S.'s, however, and as brought out in § 19.4.6, the Interaction S.S. contains error effects, and, on a hypothesis that the factors act independently, will represent pure error.

19.7.3 From the symmetry of i and j in $\sum_i\sum_j(y_{ij} - y_{i0} - y_{0j} + \bar{y})^2$, one can see that the Interaction S.S. could be equally well derived by considering the effects of phosphate applied to the i^{th} variety, viz. $y_{ij} - y_{i0}$, and their variation over the different varieties. Hence the S.S.(VP) also represents the variation in the effects of phosphate over the different varieties. In other words, as is clear from the discussion of § 19.4, "Interaction $V \times P$" and "Interaction $P \times V$" are really the same thing and are represented by the same S.S.

19.7.4 We may regard the Interaction S.S. in an alternative light. The main effects of the i^{th} variety and the j^{th} level of phosphate are, respectively, $y_{i0} - \bar{y}$ and $y_{0j} - \bar{y}$. Hence any residual

$$y_{ij} - (y_{i0} - \bar{y}) - (y_{0j} - \bar{y}) - \bar{y} = y_{ij} - y_{i0} - y_{0j} + \bar{y},$$

clearly represents a deviation of y_{ij} from the general mean after the main effects corresponding to the i^{th} and j^{th} levels of the respective factors have been deducted. The S.S. of such residuals is a measure of the extent to which the cells of the interaction table fail to coincide with expectations based on a hypothesis of additive effects (i.e. independence) of the two factors. On this hypothesis the value of y_{ij} would be estimated as $\bar{y} + (y_{i0} - \bar{y}) + (y_{0j} - \bar{y}) = y_{i0} + y_{0j} - \bar{y}$. The difference between this and y_{ij}, the actual yield, is the residual obtained above.

19.7.5 The number of D.F. allotted to the Interaction S.S. is $(m-1)(n-1)$, *and is always the product of the D.F. of the corresponding main effects.* In this example the interaction has $3 \times 2 = 6$ D.F.

19.7.6 A point which emerges from this section is that the residual or Error S.S. for a randomized blocks design is of the same form as that for interaction, and is, in fact, the interaction between treatments and blocks, i.e. a measure of the variation in treatment differences from block to block, which is usually assumed to be pure error. See, however, § 13.5.2 for a discussion of the possible non-additivity of block and treatment effects, i.e. of a genuine interaction between blocks and treatments.

19.8 Statistical model for factorial treatment effects

19.8.1 As usual a discussion in terms of a statistical model can be illuminating. Because of the position explained in § 19.3.1, we are concerned here only with a model for the treatment effects. We introduce parameters α_i to stand for the main effects of varieties (assuming these to be fixed effects), so that the expected value of the i^{th} varietal mean is $\mu + \alpha_i$, μ being a component common to all yields. Similarly, parameters β_j are introduced for the main effects (also assumed fixed) of phosphate. One linear restriction must be placed on each set of main effects, and these may be taken as $\sum_i \alpha_i = 0$ and $\sum_j \beta_j = 0$.

19.8.2 On a hypothesis of independent action of the factors (additive effects) the expected value of y_{ij} is $\mu + \alpha_i + \beta_j$, and deviations from this are due only to experimental error. On the independence hypothesis the model is therefore

$$y_{ij} = \mu + \alpha_i + \beta_j + \epsilon_{ij}, \qquad [19.2]$$

where ϵ_{ij} has zero mean and variance $\frac{\sigma^2}{r}$ (σ^2 = error variance), since y_{ij} is the mean of r replications.

19.8.3 To allow for the possibility of interaction (departures from additivity), a parameter $(\alpha\beta)_{ij}$ is introduced into the model for each treatment combination, so that the model becomes

$$y_{ij} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ij}. \qquad [19.3]$$

The $(\alpha\beta)_{ij}$ are interaction effects, assumed fixed in order to fit in with the fixed main effects, since the latter assumptions imply that the interest lies only in the particular levels of each factor actually included in the experiment. There are mn parameters $(\alpha\beta)_{ij}$, but, as indicated by the D.F. for interaction, only $(m-1)(n-1)$ of them can be linearly independent. Consequently $mn-(m-1)(n-1)=m+n-1$ linear restrictions must be imposed. It is convenient to take these as $\sum_j(\alpha\beta)_{ij}=\sum_i(\alpha\beta)_{ij}=0$, i.e. the sum of the $(\alpha\beta)_{ij}$ in any row or column of the interaction table is zero, and this preserves $\mu+\alpha_i$ and $\mu+\beta_j$ as the expected values of the marginal means whether there is interaction or not. Since there are m rows and n columns in the table, the above restrictions would seem to amount to $m+n$, but the summation of all $(\alpha\beta)_{ij}$ to zero, $\sum_i\sum_j(\alpha\beta)_{ij}=0$, is common to both sets of restrictions, the total number of which therefore reduces to $m+n-1$ as required.

*19.8.4 What the above really amounts to is that the original mn treatment parameters τ_{ij} with one restriction (or the $mn-1$ original linearly independent parameters τ_{ij}) have been replaced by the following:

Parameters	Total number	Number of restrictions	Number independent
α_i	m	1	$m-1$
β_j	n	1	$n-1$
$(\alpha\beta)_{ij}$	mn	$m+n-1$	$(m-1)(n-1)$
Total			$mn-1$

In effect the parameters τ_{ij} have been replaced by an equal number of independent parameters which are linear functions of the original τ_{ij} (in three mutually orthogonal sets). This may be called, following Kempthorne, a "reparametrization" of the model. Notice the correspondence between the number of independent parameters in each set and the number of D.F. Any orthogonal subdivision of a Treatments S.S. amounts to a reparametrization of the model in this way.

19.8.5 Model [19.3] actually represents (if we replace ϵ_{ij} by ϵ_{ijk} where k indicates replication) the model for a simple random design with two factors. For a randomized blocks or Latin square design, additional parameters representing blocks or rows and columns would have to be introduced as required. They have been deliberately omitted, partly because of the tendency to exhaust the number of suitable letters of the Greek and Roman alphabets.

19.8.6 The reason why the Interaction S.S. is so called becomes even clearer if we express any residual in terms of the model. Thus

$$y_{ij}-y_{i0}-y_{0j}+\bar{y}$$
$$=[\mu+\alpha_i+\beta_j+(\alpha\beta)_{ij}+\epsilon_{ij}]-[\mu+\alpha_i+\epsilon_{i0}]-[\mu+\beta_j+\epsilon_{0j}]+[\mu+\bar{\epsilon}]$$
(where ϵ_{i0}, ϵ_{0j}, and $\bar{\epsilon}$ have their obvious connotation)
$$=(\alpha\beta)_{ij}+(\epsilon_{ij}-\epsilon_{i0}-\epsilon_{0j}+\bar{\epsilon}),$$
i.e. each residual represents a mixture of interaction and error effects, and,

on a hypothesis of independence, only error. The main effects referred to in §§ 19.6 and 19.7 are actually only the least squares estimates of the α_i and β_j; similarly from the above $y_{ij} - y_{i0} - y_{0j} + \bar{y}$ is the estimate of $(\alpha\beta)_{ij}$.

19.9 Orthogonality of main effects and interactions

19.9.1 Since the partitioning of the Treatments S.S. is an exact subdivision, the three sets of effects in an interaction table are mutually orthogonal.

*19.9.2 However, the partitioning according to [19.1] is actually dependent on the choice of linear restrictions given in § 19.8.2, which are the natural ones to adopt when the treatment combinations are equally replicated. The orthogonality of the estimates of α_i, β_j, and $(\alpha\beta)_{ij}$ obtained on adoption of these restrictions together with those of § 19.8.1, viz. $\hat{\alpha}_i = y_{i0} - \bar{y}$, $\hat{\beta}_j = y_{0j} - \bar{y}$, and $(\widehat{\alpha\beta})_{ij} = y_{ij} - y_{i0} - y_{0j} + \bar{y}$, may be demonstrated by the method of § 13.6.12. On the other hand, if for some reason other restrictions were to be adopted, the partitioning would not be exact and the corresponding estimates would not all be orthogonal. The effect of the adoption of particular linear restrictions on the mutual orthogonality of three sets of constants was encountered previously in § 13.6.10, where it was seen that, although the adoption of the restrictions $\Sigma t_i = 0$ and $\Sigma b_j = 0$ had nothing to do with the orthogonality of the block and treatment constants in a randomized blocks design, it did ensure their orthogonality with m, the estimate of μ.

19.9.3 Should the treatment combinations not be equally replicated, orthogonality will not in general hold, and it will not be possible to obtain the Interaction S.S. by subtraction (see § 19.10.4). Cases of unequal replication therefore involve difficulties which will not be discussed further.

19.10 Statistical analysis of a two-factor experiment

19.10.1 The statistical analysis of a factorial experiment falls into two parts: (1) the basic analysis appropriate to the design (§ 19.3), (2) the partitioning of the Treatments S.S. In this respect the factorial experiment fits into the general framework outlined in § 11.7.3.

19.10.2 The same holds for the algebraic partitioning of the Total S.S. Suppose we have a randomized blocks design with r replications and that the treatments consist of a factorial arrangement with factors A (m levels) and B (n levels). If y_{ijk} represents a plot yield, where i and j denote levels of A and B and k the k^{th} block, then with the usual notation for means we have by the ordinary randomized blocks partitioning [13.2]

$$\sum_i\sum_j\sum_k(y_{ijk} - \bar{y})^2 = r\sum_i\sum_j(y_{ij0} - \bar{y})^2 + mn\sum_k(y_{00k} - \bar{y})^2 + \sum_i\sum_j\sum_k(y_{ijk} - y_{ij0} - y_{00k} + \bar{y})^2, \quad [19.4]$$

where on the R.H.S. we have S.S.'s for treatments, blocks, and error. The Treatments S.S. may now also be partitioned as in [19.1] into the S.S.'s for main effects A and B and for interaction AB, so that finally we have the complete partitioning:

$$\sum_i \sum_j \sum_k (y_{ijk} - \bar{y})^2 = \overset{\text{Blocks}}{mn \sum_k (y_{00k} - \bar{y})^2} + \overset{A}{nr \sum_i (y_{i00} - \bar{y})^2} + \overset{B}{mr \sum_j (y_{0j0} - \bar{y})^2}$$
$$+ \overset{AB}{r \sum_i \sum_j (y_{ij0} - y_{i00} - y_{0j0} + \bar{y})^2} + \overset{\text{Error}}{\sum_i \sum_j \sum_k (y_{ijk} - y_{ij0} - y_{00k} + \bar{y})^2} \quad [19.5]$$

19.10.3 Initially, therefore, if we have a randomized blocks design, the usual calculations of Blocks and Treatments S.S.'s are made (divisors being given by the respective multipliers in [19.4]), and the Error S.S. is obtained by subtraction from the Total S.S.

19.10.4 An interaction table is drawn up as in Table 19.3, except that treatment *totals* are entered in each cell, and marginal *totals* (row and column totals, Y_{i0} and Y_{0j}) are formed. The computation of the S.S.'s for the main effects then follows the general rule, the divisors being the multipliers in [19.1], and as usual representing the number of plots making up one of the relevant marginal totals. Thus for the main effect of A the S.S. is computed as

$$\frac{1}{nr}(Y_{10}^2 + \cdots + Y_{m0}^2) - C.F. \quad [19.6]$$

The Interaction S.S. is in general obtained by subtraction as

$$\text{Treatments S.S.} - \text{S.S.}(A) - \text{S.S.}(B), \quad [19.7]$$

but where a convenient method is available it should be obtained directly, in order that there will be a check on the computations.

19.11 Direct calculation of the Interaction S.S.

19.11.1 The direct calculation of the Interaction S.S., like the calculation of the Error S.S., is in general laborious; hence the subtraction method. When either or both factors are at two levels, however, a quick direct method is available and should be used.

19.11.2 *One factor at two levels.* A symbolic interaction table comprising treatment totals would be as follows:

Factor A	Factor B b_1	b_2	$\cdots$	b_n	Totals
a_1	Y_{11}	Y_{12}	$\cdots$	Y_{1n}	Y_{10}
a_2	Y_{21}	Y_{22}	$\cdots$	Y_{2n}	Y_{20}
Totals	Y_{01}	Y_{02}	$\cdots$	Y_{0n}	Y

Differences $d_1 = Y_{11} - Y_{21}$, $d_2 = Y_{12} - Y_{22}$, ... $d_n = Y_{1n} - Y_{2n}$ are formed between the totals for the levels of A at each level of B. It is immaterial whether these differences are taken as above or as $Y_{21} - Y_{11}$, $Y_{22} - Y_{12}$, etc., but the same direction of difference must be maintained for all, since differences of sign are important; usually the differences are formed so that the majority will be positive. These differences would be constant (apart from experimental error) if the effect of A were constant over all levels of B, i.e. if there were no interaction. A S.S. of the differences, viz.

$$\frac{1}{2r}({d_1}^2 + {d_2}^2 + \cdots + {d_n}^2) - \frac{(\Sigma d)^2}{2rn}, \quad [19.8]$$

therefore gives the S.S. for interaction AB. The divisor is $2r$(r = no. of replications) since each difference involves $2r$ plots. The C.F. for this S.S. is not, of course, the general C.F., and is actually (since $\Sigma d = Y_{10} - Y_{20}$) the S.S. for the main effect of A, which has 1 D.F. The interaction has $(n - 1) \times 1 = n - 1$ D.F.

19.11.3 *Both factors at two levels.* The above method can still be used, but in this case there are only two differences d_1 and d_2, and the interaction has only 1 D.F. Its S.S. can therefore be calculated most simply by the method given in § 11.2.6. The appropriate linear function which measures the interaction is the difference of the differences

$$\left.\begin{aligned} d_1 - d_2 &= (Y_{11} - Y_{21}) - (Y_{12} - Y_{22}) \\ &= (Y_{11} + Y_{22}) - (Y_{21} + Y_{12}) \\ &= \text{Difference of the diagonal totals in} \\ &\quad\ \text{the } 2 \times 2 \text{ table.} \end{aligned}\right\} \quad [19.9]$$

The Interaction S.S. is therefore

$$\frac{(Y_{11} + Y_{22} - Y_{21} - Y_{12})^2}{4r}, \quad [19.10]$$

the divisor being actually the total number of plots in the experiment.

19.11.4 Using the above methods is no advantage unless the Treatments S.S. is separately calculated and the check made:

$$\text{Treatments S.S.} = \text{S.S.}(A) + \text{S.S.}(B) + \text{S.S.}(AB)$$

19.12 Tests of significance

19.12.1 While it is common practice to draw up an analysis of variance table for the basic analysis to mark a half-way stage in the analysis, it is not really necessary to do so, since the full analysis is fore-ordained by the nature of the treatments. On rare occasions, when the Treatments S.S. is very small, it may be possible to decide that there is no point in pursuing the analysis further. Other than for this, however, there is no point in making any over-all F-test of the Treatments M.S., and usually the subdivision of the Treatments S.S. is routine.

19.12.2 Each of the M.S.'s for the main effects and interaction under Model [19.3] may be tested against the Error M.S., although the relevancy of such tests should first be considered. For instance, in the varieties × phosphate example, if there were no particular comparisons in view among the varieties, the procedure for the Varieties M.S. would be as for an unclassified sub-set (see § 11.7.2). Significance of the F-test, $\dfrac{\text{Varieties M.S.}}{\text{Error M.S.}}$, would be necessary before individual comparisons among the varietal means could legitimately be made. On the other hand the S.S. for phosphate would probably be subdivided orthogonally into the linear and quadratic effects, the M.S.'s for which would be tested against the Error M.S. The test of the M.S. for the main effect of P as a whole would be irrelevant. Similar considerations apply to the testing of the Interaction M.S. also, and this will be discussed in § 19.14.

19.12.3 The general strategy is to test the Interaction M.S. first. Should this test prove non-significant, it becomes possible to accept the simpler model [19.2], i.e. the data do not significantly controvert a hypothesis of independence of the factors. The advantage of being able to express the results of the experiment in a very simple way then accrues, since within the limits of experimental error the effects of each factor may be regarded as constant over the levels of the other factor. Such effects are estimated by the main effects obtained from the marginal means of the interaction table. For example, should the optimal treatment combination be of interest, it may be accepted as being the combination of the optimal levels of each factor as determined by the separate main effects.

19.12.4 Should the Interaction M.S. prove significant, the situation is very different and more complicated. The data are not compatible with a hypothesis that the effects of the two factors are additive. The effects of the factors in combination cannot be described solely in terms of the marginal means of the interaction table, since the effects of neither factor are consistent over the levels of the other. In other words Model [19.2] is inappropriate, and *it is necessary to examine individual treatment combinations within the body of the interaction table*; the effects of each factor have to be examined at each level of the other factor separately.

19.12.5 In the presence of interaction the main effects are likely to be of very limited interest. Thus, in the varieties-phosphate example the main effects of phosphate will have little value if the factors interact; we would wish rather to examine the effects of phosphate for each variety separately. The main effects would still be the correct average effects over the particular four varieties in the experiment, but even with varietal effects assumed fixed (indicating that interest is confined to the four varieties concerned) averages over these four varieties are unlikely to be of much interest. With the abandoning of the additive hypothesis, the main effects lose their importance in that they depend on the particular levels of the other factor selected for the experiment.

19.12.6 The only cases when the main effects of a factor (A) in the presence of interaction could be of more interest are when the levels of the second factor (B) either are the only possible levels or are representative of a population of levels. An example of the first case is when the second factor is "sex"; an average over male and female could be of interest even if the effects of A vary with sex, since it represents a population average. In the second case an assumption of random effects is made for factor B.

*19.12.7 When one of the factors, say B, is assumed to have random effects, the tests of significance are unaltered under a hypothesis of no interaction. If interaction is present, however, the main effects of A (which, as we have just seen, are of interest) are dependent on the levels of B, and since the latter are assumed to be a random selection, we must consider an additional random component in the main effect due to this source. The test of the main effect of A is therefore in these circumstances made against the Interaction

M.S., which is the estimate of the appropriate variance. A test of the main effect of B is less likely to be of interest, but, since the main effects of A are fixed, the effects of B are all meaned over the particular levels of A under study, and consequently are not subject to any additional random variance. The usual test against the Error M.S. is therefore valid. Should both A and B have random effects, then both main effects must be tested against the Interaction M.S. unless a hypothesis of independence is valid.

19.12.8 In the not very common event that the levels of each factor comprise unclassified sets of treatments, comparisons within the body of the interaction table motivated by a significant interaction may be assisted by least significant differences calculated for any two treatment combinations in the usual way. Such comparisons will mostly involve differences in the levels of one factor within any one level of the other factor, i.e. between treatment combinations in the same row or column of the interaction table, though other comparisons are not excluded. It is not, however, a question here of carrying out all possible comparisons or putting all the treatments in order of merit as with a completely unclassified set (cf. § 11.4). Usually one or both factors will have levels such that predetermined comparisons among them are appropriate, in which case there will be a corresponding subdivision of the Interaction S.S. into components, each of which is designed to provide a test of significance of relevant comparisons within the interaction table (see § 19.14).

19.13 Presentation of results in a two-factor experiment

The main feature of the presentation of results of an experiment with two factors should be an interaction table of means and marginal means in standard units together with S.E.'s and/or least significant differences for the body of the table and for both sets of marginal means. It is true that if the interaction is non-significant the marginal means are really all that is required, but it is more satisfactory to give the table as a whole.

Example 19.1 The following represents the field plan of an experiment in which five varieties of cowpeas (A, B, C, D, E) were tested in all combinations with three methods of cultivation (1, 2, 3). The yields are given in lb. per plot of $\frac{1}{100}$ morgen. Analyse the data presenting results in lb. per morgen.

	$B1$	61	$A1$	56	$E3$	62	$C1$	63	$A2$	66
Block 1	$D2$	53	$B2$	59	$D1$	65	$D3$	60	$B3$	60
	$E1$	60	$A3$	60	$C3$	65	$C2$	66	$E2$	73

	$A3$	50	$C1$	53	$E2$	77	$A2$	57	$B1$	58
Block 2	$D1$	61	$D2$	53	$C2$	58	$E3$	68	$D3$	58
	$C3$	56	$B2$	55	$E1$	61	$B3$	59	$A1$	45

Block 3	*E*3	67	*C*3	50	*C*1	49	*A*2	50	*A*3	45
	*B*2	51	*B*3	54	*E*2	77	*D*3	56	*B*1	55
	*E*1	50	*C*2	52	*D*2	48	*A*1	43	*D*1	60

Block 4	*E*1	53	*D*3	60	*E*3	60	*B*2	52	*B*1	56
	*E*2	65	*D*1	63	*D*2	55	*A*3	48	*A*1	46
	*B*3	54	*C*2	55	*C*1	48	*C*3	50	*A*2	50

Computation sheet

(A) Treatment	Block 1	Block 2	Block 3	Block 4	Treatment totals
*A*1	56	45	43	46	190
*A*2	66	57	50	50	223
*A*3	60	50	45	48	203
*B*1	61	58	55	56	230
*B*2	59	55	51	52	217
*B*3	60	59	54	54	227
*C*1	63	53	49	48	213
*C*2	66	58	52	55	231
*C*3	65	56	50	50	221
*D*1	65	61	60	63	249
*D*2	53	53	48	55	209
*D*3	60	58	56	60	234
*E*1	60	61	50	53	224
*E*2	73	77	77	65	292
*E*3	62	68	67	60	257
Block totals	929	869	807	815	3,420 (B)

C.F. = 194,940·0

Total S.S. = 198,184·0 − 194,940·0 = 3,244·0

Blocks S.S. = 195,578·4 (C) − 194,940·0 = 638·4

Treatments S.S. = 197,013·5 (D) − 194,940·0 = 2,073·5

Skeleton analysis of variance

	D.F.		D.F.
Blocks	3		
Treatments	14	Varieties	4
		Methods	2
		V × *M*	8 (E)
			14
Error	42		
Total	59		

Preliminary analysis of variance (F)

Source	D.F.	S.S.	M.S.
Blocks	3	638·4	
Treatments	14	2,073·5	
Error	42	532·1	12·67
Total	59	3,244·0	

Method of cultivation	Varieties (G) A	B	C	D	E	Method totals
1	190	230	213	249	224	1,106
2	223	217	231	209	292	1,172
3	203	227	221	234	257	1,142
Varietal totals	616	674	665	692	773	3,420 (H)

S.S.(V) = 196,029·2 (I)
194,940·0
1,089·2

S.S.(M) = 195,049·2 (J)
194,940·0
109·2

Analysis of variance

Source (K)	D.F.	S.S.	M.S.	F
Blocks	3	638·4		
Varieties (V)	4	1,089·2	272·30	21·5**
Methods of cultivation (M)	2	109·2	54·60	4·3*
Interaction (VM)	8	875·1 (L)	109·39	8·6**
Error	42	532·1	12·67	
Total	59	3,244·0		

S.E. of a single yield = 3·56

$$\text{C.V.} = \frac{3\cdot56}{3420} \times 60 \times 100 = 6\cdot25\%$$

S.E. of single treatment total (body of interaction table) = 7·12 (M)

Least significant differences (two treatment totals in body of interaction table)

$$= \sqrt{8 \times 12\cdot67} \times t\,(42\text{ D.F.})$$
$$= 10\cdot070 \times \begin{cases} 2\cdot021 \\ 2\cdot704 \end{cases} \text{(N)}$$
$$= 20\cdot4\ (5\%)$$
$$\ \ 27\cdot2\ (1\%)$$

S.E. of single varietal total = $\sqrt{12 \times 12\cdot67}$ (O) = 12·33

Least significant differences (two varietal totals) (Q) $= \sqrt{24 \times 12\cdot67} \times t\,(42\text{ D.F.})$

$$= 17\cdot44 \times \begin{cases} 2\cdot021 \\ 2\cdot704 \end{cases}$$
$$= 35\cdot2\ (5\%)$$
$$\ \ 47\cdot2\ (1\%)$$

S.E. of single method of cultivation total = $\sqrt{20 \times 12\cdot67}$ (P) = 15·92

Least significant differences (two methods of cultivations totals) (Q)

$$= \sqrt{40 \times 12\cdot67} \times t\,(42\text{ D.F.})$$
$$= 22\cdot513 \times \begin{cases} 2\cdot021 \\ 2\cdot704 \end{cases}$$
$$= 45\cdot5\ (5\%)$$
$$\ \ 60\cdot9\ (1\%)$$

Conversion factors: (R)

(1) Treatment totals to means in lb. per morg. $= \frac{100}{4} = 25$

(2) Varietal totals to means in lb. per morg. $= \frac{25}{3} = 8\cdot3\dot{3}$

(3) Methods totals to means in lb. per morg. $= \frac{25}{5} = 5$

Presentation of results

TREATMENT MEANS IN LB. PER MORGEN

Methods of cultivation	Varieties A	B	C	D	E	Means
1	4,750	5,750	5,325	6,225	5,600	5,530
2	5,575	5,425	5,775	5,225	7,300	5,860
3	5,075	5,675	5,525	5,850	6,425	5,710
Means	5,133	5,617	5,542	5,767	6,442	5,700

S.E.'s and L.S.D.'s:

	S.E.	L.S.D. (5%)	L.S.D. (1%)
Body of table	178·0	508	680
(S) Varietal means	102·8	293	393
Method means	79·6	228	304

C.V. = 6·25%.

(T) *Conclusion.* On an average over the five varieties the second cultivation method has given an increased yield of 330 lb. per morg. (significant at the 1% level) in comparison with Method 1; Method 3 has given an average yield 150 lb. per morg. less than Method 2 and 180 lb. per morg. better than Method 1, but these differences are not significant. However, the effects of the cultivation methods are not consistent over the different varieties. Thus, with Varieties *A*, *C*, and *E* Method 2 has given the best results, with increased yields over Method 1 of 825, 450, and 1,700 lb. per morg., respectively, and increased yields over Method 3 of 500, 250, and 875 lb. per morg. respectively, the least significant differences being 508 (5% level) and 680 (1% level). With varieties *B* and *D*, on the other hand, Method 1 has given the best results, with increased yields over Method 2 of 325 and 1,000 lb. per morg., respectively, and over Method 3 of 75 and 375 lb. per morg. respectively. Although only the second of these four differences is significant, it seems advisable to recommend Method 1 for use with Varieties *B* and *D*, but Method 2 with Varieties *A*, *C*, and *E*. The best combination is undoubtedly Variety *E* with Method 2, which is significantly superior to all other treatments at the 1% level, the next best being Method 3 with the same variety.

Notes on the computations

In this example both factors comprise unclassified sets and the analysis is matched accordingly to this situation. It must therefore be regarded as only a sort of basic analysis, which will have to be adapted to meet various types of classified factors (cf. Example 19.2).

(A) The usual randomized blocks analysis (15 treatments in 4 blocks) is followed, beginning with a rectangular table of blocks and treatments. The treatments are for convenience put in a systematic order, all treatments with Variety *A* being followed by all treatments with Variety *B*, etc.

(B) Check.

(C) $\frac{1}{15}(929^2 + \cdots + 815^2) - C.F.$

(D) $\frac{1}{4}(190^2 + 223^2 + \cdots + 257^2) - C.F.$

(E) See § 19.7.5. D.F. for $VM = 8 = 4 \times 2$. Check by addition.

(F) See § 19.12.1.

(G) The table is merely a rearrangement of the treatment totals. Marginal totals are computed.

(H) Check down and across to the G.T.

(I) $\frac{1}{12}(616^2 + \cdots + 773^2) - C.F.$ Divisor is 12 since each varietal total comes from 12 plots. To see this: Total no. of plots = 60; no. of varietal levels = 5; hence divisor = 60/5. Otherwise: Each treatment total comes from 4 replications, and 3 treatment totals make up a varietal total; hence divisor = 4 × 3.

(J) $\frac{1}{20}(1{,}106^2 + 1{,}172^2 + 1{,}142^2) - C.F.$ Divisor is 20 since each *M*-total comes from 20 plots (20 = 60/3 or 4 × 5—see previous note).

(K) It is usual to give an extended analysis of variance in the form shown rather than a preliminary analysis of variance together with an auxiliary subdivision of the Treatments S.S.

(L) Interaction S.S. = 2,073·5 − 1,089·2 − 109·2.

(M) $\sqrt{r \times 12 \cdot 67} = 2 \times 3 \cdot 56$ ($r = 4$).

(N) Using t for 40 D.F., which is close enough and on the conservative side.

(O) See Note I above.

(P) See Note J above.

(Q) It is usual to compute these, even though their usefulness is limited (see § 19.12.5).

(R) Three conversion factors are necessitated by the decision to work out the S.E.'s and L.S.D.'s in terms of treatment totals. The alternative would be to work these out in terms of means, in which case there would be one conversion factor, for means in experimental units to means in standard units.

(S) In this table the first, second, and third rows are obtained by applying the first, second, and third conversion factors, respectively.

(T) *Had there been no significant evidence of interaction* the discussion could have been entirely in terms of the marginal means, e.g.:

> "Variety *E* has yielded significantly higher than all other varieties, the next best being Variety *D* with a mean yield of 675 lb. per morg. less (significant at 1% level). Variety *A* has yielded significantly lower than all other varieties, the next worst (*B*) having a yield 484 lb. per morg. better (significant at 1% level). There are no significant differences between Varieties *B*, *C*, and *D*.
>
> As regards methods of cultivation, Method 2 has given an increased yield of 330 lb. per morg. (significant at the 1% level) in comparison with Method 1. Method 3 is intermediate and its differences in yield from Methods 1 and 2 (viz. +180 lb. per morg. and −150 lb. per morg.) are not significant.
>
> The above statement with respect to varieties is applicable irrespective of method of cultivation, and the differences between methods of cultivation are consistent for all the varieties studied. The recommended treatment combination is therefore Variety *E* with Method 2."

Notice that in this example the interaction has not actually affected the conclusion concerning the optimum combination based on the marginal means alone as if no interaction were present.

19.14 Conformable partitioning of the Interaction S.S.

19.14.1 Corresponding to any orthogonal partitioning of the S.S. of either main effect there exists a parallel or **conformable partitioning** of the Interaction S.S. into orthogonal components. Hence, when the levels of either factor comprise a classified set such that an orthogonal partitioning of the S.S. for the main effect of that factor is called for, the conformable partitioning of the Interaction S.S. should also be carried out. The exception is when the Interaction S.S. is so small that no component M.S. could possibly be significant. Partitioning the Interaction S.S. in this way may show up significant interaction components which could remain undetected in the over-all test of the Interaction M.S. Attention is also drawn to comparisons within the interaction table which are relevant in the light of the nature of the factors, and the appropriate tests of significance can be performed. *Significance of any component of the interaction will, of course, imply that a hypothesis that the factors act independently is not tenable.*

19.14.2 There are two possible cases: (i) one factor classified, the other unclassified; (ii) both factors classified. As an example of the former, which is the simpler of the two, we turn once more to the varieties-phosphate example of § 19.1.3, the varieties being assumed to comprise an unclassified set and the levels of phosphate being assumed equally spaced. The main effect of phosphate (P) will usually be orthogonally partitioned as P' (linear effect) and P'' (quadratic effect), these being the components of interest, and the process will be facilitated by use of the appropriate orthogonal polynomial coefficients, viz. $-1 \quad 0 \quad 1$ (linear), and $1 \quad -2 \quad 1$ (quadratic).

19.14.3 Before conclusions may be drawn on the basis of these main effect components, there must be some assurance that V and P do not interact. The general test of VP may show non-significance, but this is not necessarily conclusive; alternatively, VP may be significant, which certainly causes the additive hypothesis to be untenable, but additional information on the nature of the interaction can be obtained which may be valuable in the interpretation of the results.

19.14.4 Since P' and P'' are the components of P which are of interest, it is necessary to study how they vary over the different varietal levels, i.e. to study the interactions VP' and VP'', which are components of VP. To do this, values of P' are calculated by applying the orthogonal polynomial coefficients $-1 \quad 0 \quad 1$ to $p_0 \quad p_1 \quad p_2$ *at each level of* V *separately*. In this way we obtain the following quantities (using the notation of Table 19.1):

Table 19.4: Values of P' (symbolic) at the different levels of V

	$P'v_i$	Linear function
	$P'v_1$	$-v_1p_0 + v_1p_2$
	$P'v_2$	$-v_2p_0 + v_2p_2$
	$P'v_3$	$-v_3p_0 + v_3p_2$
	$P'v_4$	$-v_4p_0 + v_4p_2$
Totals	P'	$-p_0 + p_2$

(N.B.: v_ip_j here represents the *total* yield of all plots with this treatment combination; p_i *on its own* represents the total of all plots which receive the i^{th} level of P as part of its treatment combination.)

Here $P'v_i$ indicates the value of P' at the i^{th} level of V. The total of these over all levels of V, viz. $\sum_i P'v_i$, is P', the over-all linear effect of P. Notice that the symbolic representation of this as a linear function $(-p_0 + p_2)$ is quite unambiguous here, but could be ambiguous if the type of notation envisaged in § 19.2.3 were to be used.

19.14.5 Variation in P' over the different levels of V may be assessed by computing a S.S. of the $P'v_i$ (obtained from treatment *totals*) about their own mean, viz.

$$\frac{1}{r\Sigma\lambda^2}\Sigma(P'v_i)^2 - \frac{(P')^2}{mr\Sigma\lambda^2}, \qquad [19.11]$$

where $\Sigma\lambda^2$ is the S.S. of the coefficients used in obtaining P' (here 2) and m is the number of levels of V (here 4). The C.F. for this S.S., obtained on the usual principle of squaring the total of the individual values (here P') by a quantity obtained as the product of the divisor $(r\Sigma\lambda^2)$ and the number of individual values (m), is actually the S.S.(P'). The latter will usually have been obtained in the subdivision of P, and, of course, need not be computed again. The S.S. for VP' obtained by [19.11] has 3 D.F., since there are 4 values $P'v_i$; alternatively, P' has 1 D.F., V has 3 D.F., and the product rule (§ 19.7.5) still applies.

19.14.6 An exactly similar computation may be made to obtain the S.S.

for VP''. An evaluation is made of $P''v_i = v_ip_0 - 2v_ip_1 + v_ip_2$ for all levels of V, and a S.S. is calculated in the same way as in [19.11], where the C.F. will now be the S.S.(P''). The S.S.(VP'') also has 3 D.F., and we thus have an orthogonal partitioning of VP on the following lines:

	D.F.
VP'	3
VP''	3
VP	6

19.14.7 In the event of an interaction component proving significant, say VP', an examination of the numerical quantities in Table 19.4 will show where the variability occurs. It could happen that the over-all linear effect could show no significant linear response, but that the interaction could be such that with some varieties a marked positive response is obtained and with others a negative response. More likely in the varieties-phosphate example is the situation where there is an over-all response, but that it is much more marked with some varieties than with others. Then from Table 19.4 we could work out the linear responses for each variety separately, should a significant interaction VP' necessitate this.

19.14.8 The main interest in P'' and VP'' lies in the fact that the non-significance of both will permit the results to be expressed as a common linear response to phosphate (if VP' is non-significant), or as a set of individual linear responses for each variety (if VP' is significant). Significance of either P'' or VP'' will show that significant deviations from linearity occur, and it will not be very much help if P'' is significant and VP'' is non-significant, indicating that there is a more or less constant curvature of the response curve for all varieties. On the other hand one can no longer depend on P'' for a test of linearity; the over-all quadratic effect could quite easily be non-significant and yet conceal significantly different curvatures for the various varieties.

19.14.9 The second case, where both factors comprise classified sets, may be illustrated by assuming some sort of classification for V above. Assume, for example, that v_1 represents a standard variety and the remainder new varieties. It is now of interest to know whether the variation in P' over the different varieties (i.e. VP') bears any relationship to the varietal classification. The S.S.(V) will, of course, be subdivided as

	D.F.
Standard v. new varieties	1
New varieties amongst themselves	2
Varieties	3

in the manner explained in Chapter 11. If now exactly the same subdivision in exactly the same manner is carried out on the quantities of Table 19.4, except that the divisors used in subdividing V should be multiplied by $\frac{\Sigma\lambda^2}{n}$, we will have made a comformable partitioning of VP' as:

	D.F.
(Standard *v.* new varieties) $\times P'$	1
(New varieties amongst themselves) $\times P'$	2
VP'	3

Significance of the first of these components will indicate that the linear response to phosphate is not the same for the standard variety as it is on the average for the new varieties; significance of the second component will indicate that the linear response to phosphate is not constant over the new varieties. Significance of either will indicate that the linear response to phosphate is not constant over varieties as a whole, but in view of the classification of varieties, this is hardly relevant any longer.

A similar partitioning of VP'' can also be made.

19.14.10 In the example just considered it is natural to choose some particular linear function of P and to examine its variation over the different varieties, but there are many instances where it is impossible or not easy to differentiate between the factors in this way. For example, suppose we have two factors P and K, phosphate and potash, each at three equally spaced levels designated by p_0, p_1, p_2 and k_0, k_1, k_2. Table 19.4 then becomes

$$\begin{aligned} P'k_0 &= -p_0k_0 + p_2k_0 \\ P'k_1 &= -p_0k_1 + p_2k_1 \\ P'k_2 &= -p_0k_2 + p_2k_2 \\ \hline P' &= -p_0 + p_2 \ . \end{aligned}$$

It is now desirable to examine exactly how P' varies over the different levels of K (the situation being similar to that of the example in § 19.14.9), and, seeing that the levels of K are also equally spaced, the obvious procedure here will be to apply the same coefficients to $P'k_0$, $P'k_1$, and $P'k_2$ as are applied to k_0, k_1, and k_2 to obtain K' and K''. We therefore obtain

$$-P'k_0 + P'k_2 = p_0k_0 - p_2k_0 - p_0k_2 + p_2k_2 \qquad [19.12]$$

and

$$P'k_0 - 2P'k_1 + P'k_2 = p_2k_2 - p_0k_2 - 2p_0k_1 + 2p_2k_1 + p_2k_0 - p_0k_0. \qquad [19.13]$$

19.14.11 Similarly we may obtain

$$\begin{aligned} P''k_0 &= p_0k_0 - 2p_1k_0 + p_2k_0 \\ P''k_1 &= p_0k_1 - 2p_1k_1 + p_2k_1 \\ P''k_2 &= p_0k_2 - 2p_1k_2 + p_2k_2 \\ \hline P'' &= p_0 - 2p_1 + p_2 \end{aligned}$$

and, applying the coefficients $-1 \quad 0 \quad 1$ and $1 \quad -2 \quad 1$ to $P''k_0$, $P''k_1$, and $P''k_2$, we obtain

$$-P''k_0 + P''k_2 = p_2k_2 - 2p_1k_2 + p_0k_2 - p_2k_0 + 2p_1k_0 - p_0k_0 \qquad [19.14]$$

and

$$P''k_0 - 2P''k_1 + P''k_2$$
$$= p_0k_0 - 2p_1k_0 + p_2k_0 - 2p_0k_1 + 4p_1k_1 - 2p_2k_1 + p_0k_2 - 2p_1k_2 + p_2k_2. \qquad [19.15]$$

19.14.12 However, there is no reason why we should not have formed the quantities $K'p_0$, $K'p_1$, $K'p_2$ and $K''p_0$, $K''p_1$, $K''p_2$ and applied the orthogonal polynomial coefficients to these, in which case it would have been found that we arrived at the same linear functions [19.12], [19.13], [19.14], and [19.15], which are denoted as $P'K'$, $P'K''$, $P''K'$, and $P''K''$ respectively and are termed the linear × linear, linear × quadratic, quadratic × linear, and quadratic × quadratic interaction components, respectively. This is another example of the symmetrical nature of interaction in relation to the factors concerned (cf. §§ 19.4.1 and 19.7.3).

19.14.13 The interpretation of these components should be clear. Thus $P'K'$ is a measure of the extent to which the linear response to P changes linearly over the levels of K. For example, if $P'k_0 = 20$, $P'k_1 = 30$, $P'k_2 = 40$, then the linear response to phosphate is changing linearly (i.e. in equal steps) over the levels of K. In the light of § 19.14.12, it is equally a measure of the extent to which the linear response to K changes linearly over the levels of P. This is likely to be the most important of these four components.

Similarly, $P'K''$ is a measure of the extent to which the linear response to P follows a quadratic curve (i.e. departs from linearity) as the levels of K change. For example, if $P'k_0 = 20$, $P'k_1 = 30$, $P'k_2 = 20$, then P' is not changing linearly over the levels of K. Equally, $P'K''$ is a measure of the extent to which the quadratic response to K changes linearly over the levels of P, as would be the case if we had $K''p_0 = -4$, $K''p_1 = 2$, $K''p_2 = 8$.

19.14.14 The linear functions [19.12], [19.13], [19.14], and [19.15] may be derived in a somewhat more direct way as follows. Consider the linear function P' obtained by applying the coefficients $-1 \quad 0 \quad 1$, not to the marginal totals for levels of P, but to the treatment totals, i.e. to each individual treatment combination, thus:

p_0k_0	p_1k_0	p_2k_0	p_0k_1	p_1k_1	p_2k_1	p_0k_2	p_1k_2	p_2k_2
−1	0	1	−1	0	1	−1	0	1

This obviously amounts to the same thing, since p_0 gets the coefficient −1 wherever it appears, p_1 gets the coefficient 0, and so on (cf. § 11.2.4 for a similar switch). Let us now for convenience introduce x to stand for the vector of treatment totals in the order given above. Then in the notation introduced in § 10.3.10 (cf. also § 13.6.12) we may write

$$\begin{aligned}
P' &= [-1 \quad 0 \quad 1 \qquad -1 \quad 0 \quad 1 \qquad -1 \quad 0 \quad 1]x \\
P'' &= [1 \quad -2 \quad 1 \qquad 1 \quad -2 \quad 1 \qquad 1 \quad -2 \quad 1]x \\
K' &= [-1 \quad -1 \quad -1 \qquad 0 \quad 0 \quad 0 \qquad 1 \quad 1 \quad 1]x \\
K'' &= [1 \quad 1 \quad 1 \qquad -2 \quad -2 \quad -2 \qquad 1 \quad 1 \quad 1]x,
\end{aligned}$$

all of which are, of course, mutually orthogonal, since P and K as a whole are orthogonal and we have orthogonal subdivisions of each.

19.14.15 If now we multiply corresponding coefficients of any two of these vectors, we obtain a vector whose coefficients must sum to zero (by the orthogonality rule [11.6]) and which therefore constitutes a comparison among the treatments (see § 11.2.1). If we choose one vector from the P subdivision and one vector from the K subdivision, the new vector so obtained will also be found to be orthogonal not only to all the above vectors but also to all the other three vectors which can be obtained in this way. If we choose P' and K', for example, we obtain

$$[1 \quad 0 \; -1 \qquad 0 \quad 0 \quad 0 \qquad -1 \quad 0 \quad 1]x.$$

But this is the same as the linear function [19.12], and is therefore $P'K'$.

19.14.16 In the same way we can reproduce the vectors $P'K''$, $P''K'$, $P''K''$ corresponding to the linear functions [19.13], [19.14], and [19.15]. We thus have a complete orthogonal subdivision of the 4 D.F. for the interaction PK, and, together with the complete orthogonal subdivisions of P and K given in § 19.14.14, a complete orthogonal subdivision of the Treatments S.S.

19.14.17 An even simpler method is to "multiply" the linear functions of the main effect components; e.g.

$$P'K' = (-p_0 + p_2)(-k_0 + k_2) = p_0k_0 - p_2k_0 - p_0k_2 + p_2k_2.$$

This rule even holds for the basic divisors, $\Sigma\lambda^2$. For P' the basic divisor is 2, and the same is true for K'. Hence the basic divisor for $P'K'$ is $2 \times 2 = 4$.

19.14.18 S.S.'s for the four interaction components each with one D.F. are obtained in the usual way as the square of the linear function divided by the quantity given by [11.3]. Each linear function could be tested by the t-test, but here in view of the complete orthogonal subdivision the check provided by the S.S.'s is valuable.

19.14.19 It may now be realized that the *product notation for the interaction of two factors* (e.g. $P \times K$) has to some extent a literal justification, if we consider the multiplication of D.F. (§ 19.7.5) and now the "multiplication" of vectors corresponding to main effect components to give an interaction component.

*19.14.20 Just as the obtaining of linear, quadratic, cubic, . . . effects for a single factor corresponds to the fitting of a polynomial curve for y (yield) in terms of x (level of factor), viz.

$$y = a + bx + cx^2 + dx^3 + \cdots,$$

so the obtaining of linear $\times$ linear, linear $\times$ quadratic, etc. responses in conjunction with the linear and quadratic components of the main effects corresponds to the estimation of an expression for y in terms of x_1 (level of one factor) and x_2 (level of the second factor) in the form

$$y = a + bx_1 + cx_2 + dx_1^2 + ex_1x_2 + fx_2^2 + gx_1^2x_2 + hx_1x_2^2 + ix_1^2x_2^2.$$

This is a **response surface** and may be used in a similar way (but, of course, the application is more complicated) to the response curves for single factors.

Example 19.2 Given that in Example 19.1 the factor called "methods of cultivation" was in fact a factor "spacing in the rows" (S) at three levels 4″, 8″, and 12″ (corresponding to Methods 1, 3, and 2, respectively), analyse the data accordingly. (Data from Saunders & Rayner, *Statistical methods with special reference to field experiments*)

Computation sheet (A)

Spacing	Varieties A	B	C	D	E	Totals
0	190	230	213	249	224	1,106
(B) 1	203	227	221	234	257	1,142
2	223	217	231	209	292	1,172
Totals	616	674	665	692	773	3,420
(C) $-s_0 + s_2$ (S')	33	−13	18	−40	68	66
(D) $s_0 - 2s_1 + s_2$ (S'')	7	−7	2	−10	2	−6 (E)

S.S.(S') = 108·9 (F)
S.S.(S'') = 0·3
Total 109·2 = S.S.(S)

(G) S.S.(VS') $= \frac{1}{8}(33^2 + 13^2 + \cdots + 68^2) -$ S.S.(S')
= 975·8
108·9
866·9

(H) S.S.(VS'') $= \frac{1}{24}(7^2 + 7^2 + \cdots + 2^2) -$ S.S.(S'')
= 8·6
0·3
8·3

S.S.(VS') + S.S.(VS'') = 875·2 = S.S.(VS)

Auxiliary analysis of variance (I)

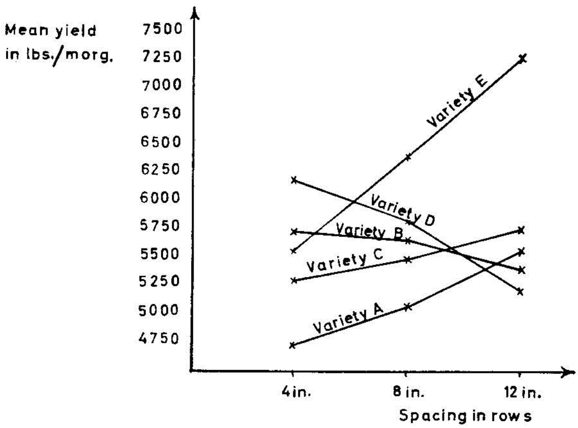
Source	D.F.	S.S.	M.S.	F
S'	1	108·9	108·9	8·6**
S''	1	0·3	0·3	N.S.
S	2	109·2		
VS'	(J) 4	866·9	216·7	**
VS''	4	8·3	2·1	N.S.
VS	8	875·2		
Error	42		12·67	

Conclusion. Increasing the spacing in the rows has resulted in an average increase in yield of 165 (K) ± 56·3 lb. per morg. for each increase of 4″ between plants in the row. This increase, although apparently consistent over the range of spacings studied, is, however, not consistent from variety to variety. Figures for the average increase for the separate varieties are as follows:

Variety	Average response in lb. per morg. to increasing row spacing by 4″	S.E.
A	412 (L)	125·9 (M)
B	−162	
C	225	
D	−500	
E	850	
Mean	165	56·3

This table reveals, as the diagram below graphically illustrates, that the increase of spacing actually decreases the yields with Varieties B and D, although it increases the yields with the other varieties. Of these responses those with Varieties A, D, and E are significant at the 1% level (N), and there is no reason to consider that for any variety the response varies over the range of spacings studied. (O)

It is therefore recommended that with Varieties A, C, and E the 12″ spacing be used, but with Varieties B and D the 4″ spacing. The best combination is Variety E with the 12″ spacing, which is significantly superior to all other combinations.



Notes on the computations

(A) To save space an abridged computation sheet only is presented. The computation sheet for Example 19.1 serves as a basic analysis, and only the conclusions need to be deleted. The only amendments necessary are in respect of the change of notation from M to S and the change of order of the levels of this factor. These computations will therefore not be presented again. The interaction table would probably, however, have certain additions to it and so is reproduced in its amended form with these additions. Here V is an unclassified factor and S a classified factor. The analysis will therefore be on the lines discussed in §§ 19.14.2 to 19.14.8.

(B) The levels of S have been designated 0, 1, 2 to fit in with the symbolic discussion of this chapter. Hence 0 = 4″ spacing in rows, 1 = 8″, 2 = 12″.

(C) Since S has equally spaced levels, we apply the appropriate orthogonal polynomial coefficients $-1 \quad 0 \quad 1$ to $s_0 \quad s_1 \quad s_2$ for each variety separately, as in Table 19.4.

(D) Similarly we apply $1 \; -2 \quad 1$ to obtain the quadratic effects for each variety separately.

(E) The totals of these rows give linear functions for S' and S'', and should be checked as $-1106 + 1172$ and $1106 - 2(1142) + 1172$, respectively, from the marginal totals.

(F) $\text{S.S.}(S') = \dfrac{(66)^2}{2 \times 20}$ (each spacing total comprises 20 plots),

$$\text{S.S.}(S'') = \frac{6^2}{6 \times 20},$$

and they sum to the S.S.(S) = 109—formerly the S.S.(M)—apart from a rounding error.

(G) In deference to the slightly increased difficulty of these S.S.'s, the computational formulae have, contrary to usual practice, been included in the computation sheet. Formula [19.11] is used, with $\Sigma\lambda^2$ here = 2, $r = 4$, $m = 5$.

(H) This is very similar, but $\Sigma\lambda^2$ now = 6. Hence divisor = 6 × 4.

(I) If desired one grand analysis of variance table could be produced, but this is by no means necessary.

(J) There are 5 values each of $S'v_i$ and $S''v_i$, since V has 5 levels. Hence VS' and VS'' each has 4 D.F.

(K) $165 = \dfrac{66}{2} \times 5$

$$= \frac{S' \text{ (from spacing totals)}}{\Sigma\lambda^2} \times \text{conversion factor (3)}$$

$$\text{S.E.} = \sqrt{2 \times 20 \times 12{\cdot}67} \times \tfrac{5}{2} = \frac{1}{\sqrt{2}}(79{\cdot}6),$$

where 79·6 is the S.E. for a single spacing mean in lb. per morg. as obtained in Example 19.1. As a check, $t^2 = \left(\dfrac{165}{56{\cdot}3}\right)^2 = (2{\cdot}93)^2 = 8{\cdot}6 = F$.

(L) $412 = \frac{33}{2} \times 25$

$= \frac{S'v_A}{2} \times$ conversion factor (1).

(M) $125 \cdot 9 = \frac{1}{\sqrt{2}} (178 \cdot 0)$, where $178 \cdot 0 =$ S.E. for a single treatment mean in lb. per morg. as obtained in Example 19.1.

(N) Least significant levels are given by $125 \cdot 9 \times t = 125 \cdot 9 \times \begin{cases} 2 \cdot 021 \\ 2 \cdot 705 \end{cases} = \begin{matrix} 254\ (5\%) \\ 340\ (1\%). \end{matrix}$

(O) Because neither S'' nor $S''V$ is significant, or approaches significance.

19.15 Experiments with three factors

19.15.1 The introduction of a third factor into a treatment arrangement constitutes a simple extension of the two-factor scheme. If in addition to varieties (4 levels) and phosphate (3 levels), a factor potash with 2 levels (k_0 and k_1) is introduced, the treatments will consist of all possible combinations, 24 in all, of one level from each of the three factors, which might be represented by the following notation:

$v_1p_0k_0$	$v_1p_0k_1$	$v_2p_0k_0$	$v_2p_0k_1$	$v_3p_0k_0$	$v_3p_0k_1$	$v_4p_0k_0$	$v_4p_0k_1$
$v_1p_1k_0$	$v_1p_1k_1$	$v_2p_1k_0$	$v_2p_1k_1$	$v_3p_1k_0$	$v_3p_1k_1$	$v_4p_1k_0$	$v_4p_1k_1$
$v_1p_2k_0$	$v_1p_2k_1$	$v_2p_2k_0$	$v_2p_2k_1$	$v_3p_2k_0$	$v_3p_2k_1$	$v_4p_2k_0$	$v_4p_2k_1$

19.15.2 More generally, if we have three factors A, B, and C with m, n, and p levels respectively, the partitioning of the Treatments S.S. may be outlined by the following skeleton analysis of variance:

Table 19.5: Partitioning of D.F. for a three-factor experiment

Source	D.F.
Main effect of A	$m - 1$
Main effect of B	$n - 1$
Main effect of C	$p - 1$
Interaction AB	$(m - 1)(n - 1)$
Interaction AC	$(m - 1)(p - 1)$
Interaction BC	$(n - 1)(p - 1)$
Interaction ABC	$(m - 1)(n - 1)(p - 1)$
Total treatments	$mnp - 1$

The appearance of S.S.'s for the interactions AB, AC, and BC, is logical, since allowance must be made for the possibility that any pair of factors may interact, but we are left with a residual S.S. which has been labelled "Interaction ABC". What sort of treatment effect is represented by this component of the Treatments S.S.?

19.16 Second order interactions

19.16.1 To obtain some idea of what is meant by the interaction ABC, let us consider the three 2×2 tables of Table 19.2 as representing the interaction of two factors A and B (each at two levels) at the three levels of a third factor C:

Table 19.6: Numerical example of a second order interaction

	b_0	b_1
a_0	12	16
a_1	15	19

Level c_0

	b_0	b_1
a_0	12	16
a_1	15	25

Level c_1

	b_0	b_1
a_0	12	16
a_1	15	13

Level c_2

Thus 19 represents the true yield of the treatment combination $a_1b_1c_0$.

19.16.2 When these same figures were discussed previously, it was found that these three tables illustrated no interaction (level c_0) and interactions in different directions (levels c_1 and c_2), in the sense that at level c_1 the yield of a_1b_1 is greater than its expectation (19) on the basis of additivity, while at level c_2 it is below expectation. In fact, from [19.9] and [19.10], we see that the interaction of A and B at the different levels of C can be represented in each case by the difference of the two diagonal totals, viz.:

Level of C	c_0	c_1	c_2
Value of interaction AB	0	6	−6

What is clear is that the interaction AB does not stay the same over the three levels of C, and in these circumstances there is said to be an interaction between AB and C, referred to as the interaction $AB \times C$ or ABC. An interaction such as AB is called a **first order interaction** or **two-factor interaction.** An interaction such as ABC is called a **second order interaction**, a **three-factor interaction**, or sometimes a triple interaction.

19.16.3 In a three-factor arrangement the interaction of any pair of factors (first order interaction) must be examined by means of the relevant interaction table, summed or meaned (depending on whether we are using treatment totals or treatment means) *over all levels of the third factor*. Thus in the example just discussed the interaction table of true mean yields for A and B would be obtained by meaning the three 2×2 tables of Table 19.6. This gives

	b_0	b_1
a_0	12	16
a_1	15	19 ,

the same as for level c_0, and represents zero interaction AB. If we were to judge from the two-factor table only, therefore, we would conclude that the factors A and B act independently, but as we find from an examination of AB at the separate levels of C, this is not so in the presence of c_1 and c_2. The necessity for studying the second order interaction is therefore apparent.

19.16.4 Table 19.6 is an example of a **three-factor interaction table**, and in general such a table consists of tables such as Table 19.1 for the factor-pair AB constructed separately for each level of the third factor C. If the interaction AB varies over the different levels of C, this constitutes a second order interaction ABC, but the existence of ABC does not depend on the existence of AB. For example, if A and B are independent at all levels of C, there is no interaction ABC, but, if A and B interact in exactly the same way at all levels of C, there is also no interaction ABC, since AB does not vary with C. In practice, however, as opposed to theory, we usually encounter second order interactions when the pairs of factors themselves interact, rather than the situation represented by Table 19.6. In any case, if either AB or ABC exists, or both, the factors A and B are not independent in their effects.

19.16.5 There is the usual complication in that apparent second order interactions can be observed which merely represent experimental error and not genuine treatment effects. This difficulty is resolved by the appropriate test of significance.

19.17 Algebraic partitioning of the Treatments S.S. in a three-factor arrangement

19.17.1 The Total S.S. is first partitioned according to whatever basic design has been used. This section concerns the orthogonal subdivision of the Treatments S.S. so obtained.

19.17.2 Let the treatment means over all replications be y_{ijk} where i, j, and k represent levels of factors A, B, and C at m, n, and p levels respectively, and let there be r replications of each treatment combination. Then by [19.5] we have

$$r\sum_i\sum_j\sum_k(y_{ijk}-\bar{y})^2 = npr\sum_i(y_{i00}-\bar{y})^2 + mpr\sum_j(y_{0j0}-\bar{y})^2 + mnr\sum_k(y_{00k}-\bar{y})^2$$
$$+ pr\sum_i\sum_j(y_{ij0}-y_{i00}-y_{0j0}+\bar{y})^2 + r\sum_i\sum_j\sum_k(y_{ijk}-y_{ij0}-y_{00k}+\bar{y})^2,$$

where the terms on the R.H.S. now represent, in order, main effects of A, B, and C, interaction AB, and residuals.

Applying [10.15] to the residual S.S., we have for particular values of i and k

$$r\sum_j(y_{ijk}-y_{ij0}-y_{00k}+\bar{y})^2 = nr(y_{i0k}-y_{i00}-y_{00k}+\bar{y})^2$$
$$+ r\sum_j(y_{ijk}-y_{ij0}-y_{i0k}+y_{i00})^2,$$

and for all values of i and k

$$r\sum_i\sum_j\sum_k(y_{ijk}-y_{ij0}-y_{00k}+\bar{y})^2 = nr\sum_i\sum_k(y_{i0k}-y_{i00}-y_{00k}+\bar{y})^2$$
$$+ r\sum_i\sum_j\sum_k(y_{ijk}-y_{ij0}-y_{i0k}+y_{i00})^2, \quad [19.16]$$

where the first S.S. on the R.H.S. is the S.S. for the interaction AC.

The remaining S.S. is now partitioned similarly. For particular values of j and k we have

$$r\sum_i(y_{ijk}-y_{ij0}-y_{i0k}+y_{i00})^2 = mr(y_{0jk}-y_{0j0}-y_{00k}+\bar{y})^2$$
$$+ r\sum_i(y_{ijk}-y_{ij0}-y_{i0k}-y_{0jk}+y_{i00}+y_{0j0}+y_{00k}-\bar{y})^2,$$

and for all values of j and k

$$r\sum_i\sum_j\sum_k(y_{ijk}-y_{ij0}-y_{i0k}+y_{i00})^2 = mr\sum_j\sum_k(y_{0jk}-y_{0j0}-y_{00k}+\bar{y})^2$$
$$+ r\sum_i\sum_j\sum_k(y_{ijk}-y_{ij0}-y_{i0k}-y_{0jk}+y_{i00}+y_{0j0}+y_{00k}-\bar{y})^2.$$

The first S.S. on the R.H.S. is the S.S. for the interaction BC, and the remaining S.S. must therefore be the S.S. for the interaction ABC, so that the partitioning is now complete.

19.18 Examination of the S.S. for the second order interaction

19.18.1 A second order interaction ABC has been defined as a measure of the variation in the first order interaction AB over the different levels of C. We saw that for any given levels i and j of A and B the interaction AB was measured by $y_{ij0} - y_{i00} - y_{0j0} + \bar{y}$. At a particular level of C, say the k^{th}, the interaction AB is measured by $y_{ijk} - y_{i0k} - y_{0jk} + y_{00k}$, with mean over all levels of C equal to $y_{ij0} - y_{i00} - y_{0j0} + \bar{y}$. Consequently, the S.S. of these quantities about their mean, viz.

$$\sum_k[(y_{ijk} - y_{i0k} - y_{0jk} + y_{00k}) - (y_{ij0} - y_{i00} - y_{0j0} + \bar{y})]^2$$
$$= \sum_k(y_{ijk} - y_{ij0} - y_{i0k} - y_{0jk} + y_{i00} + y_{0j0} + y_{00k} - \bar{y})^2,$$

represents the variation of the interaction AB (for a particular pair of levels of A and B) over the levels of C. Summing over all levels of A and B by taking $\sum_i\sum_j$, we arrive at the final S.S. of the partitioning of § 19.17.2 (apart from the multiplier r for the number of replications). This S.S. therefore represents the interaction $AB \times C$ or ABC.

19.18.2 The symmetry of the suffices i, j, and k in the expression for the S.S.(ABC) shows that it could have been similarly derived by considering the variation of the interaction AC over the different levels of B, or the variation in the interaction BC over the different levels of A. Hence the interactions $AB \times C$, $AC \times B$, and $BC \times A$ are all the same thing, denoted by ABC. Similarly, as regards the construction of a second order interaction table, in addition to the arrangement explained in § 19.16.4, one could equally well construct $A \times C$ tables for each level of B, or $B \times C$ tables for each level of A.

19.18.3 The number of D.F. for ABC is

$$mnp - 1 - (m-1) - (n-1) - (p-1) - (m-1)(n-1)$$
$$- (m-1)(p-1) - (n-1)(p-1) = (m-1)(n-1)(p-1),$$

i.e. *the product of the D.F. for A, B, and C.*

19.18.4 The statistical model for the treatment effects in a three-factor experiment may be taken as

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk}, \quad [19.17]$$

where α_i, β_j, γ_k are fixed main effect components with $\Sigma\alpha_i = \Sigma\beta_j = \Sigma\gamma_k = 0$, and $(\alpha\beta)_{ij}$, $(\alpha\gamma)_{ik}$, $(\beta\gamma)_{jk}$ are fixed first order interaction components with $\sum_i(\alpha\beta)_{ij} = \sum_j(\alpha\beta)_{ij} = \sum_i(\alpha\gamma)_{ik} = \sum_k(\alpha\gamma)_{ik} = \sum_j(\beta\gamma)_{jk} = \sum_k(\beta\gamma)_{jk} = 0$ (cf. § 19.8.3). The triple interaction components must have $mnp - (m-1)(n-1)(p-1)$ linear restrictions imposed and these may be taken as

$$\sum_i(\alpha\beta\gamma)_{ijk} = \sum_j(\alpha\beta\gamma)_{ijk} = \sum_k(\alpha\beta\gamma)_{ijk} = 0.$$

On a hypothesis that the triple interaction is zero, viz. that the factors interact only in pairs, the last component is dropped from the model. For the sake of simplification it is usually hoped that non-significance of ABC will permit this, and in practice second order interactions tend to be of less importance than those of the first order.

19.18.5 From the contents of the brackets in the partitioning of the Treatments S.S. we can see that the second order interaction residual $y_{ijk} - y_{ij0} - y_{i0k} - y_{0jk} + y_{i00} + y_{0j0} + y_{00k} - \bar{y}$ may be written as $y_{ijk} - \bar{y} - (y_{i00} - \bar{y}) - (y_{0j0} - \bar{y}) - (y_{00k} - \bar{y}) - (y_{ij0} - y_{i00} - y_{0j0} + \bar{y}) - (y_{i0k} - y_{i00} - y_{00k} + \bar{y}) - (y_{0jk} - y_{0j0} - y_{00k} + \bar{y})$, i.e. the residual represents a deviation of y_{ijk} from the general mean after deduction of the three main effects and the three first order interaction deviations, i.e. the second order interaction is a measure of the extent to which the cells of the three-factor interaction table fail to coincide with expectations based on a hypothesis that the factors interact only in pairs.

19.18.6 It is a not uncommon error to think that a second order interaction ABC occurs when the factor C has effects in the presence of A and B greater than would be expected on the basis of the effects of the same levels of A, B, and C separately. Actually, from the model, this would be

$$y_{ijk} - (\mu + \alpha_i + \beta_j + \gamma_k) = (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk},$$

and this clearly reflects first order interactions as well as the second order interaction.

19.19 Statistical analysis of three-factor experiment

19.19.1 There is little that is new about the calculations of the S.S.'s. Following the preliminary analysis in accordance with the design used, the components of the Treatments S.S. are obtained. S.S.'s for all main effects and first order interactions are obtained by drawing up interaction tables for every possible pair of factors, and then proceeding on the lines explained in § 19.10 or § 19.11. There are three points to bear in mind here:

(1) The two-factor interaction tables must contain treatment totals *taken over all levels of the third factor.*

(2) In consequence of this, *the symbolic divisors shown in* [*19.6*], [*19.8*], *and* [*19.10*] *must be multiplied by the number of levels of the third factor*, since each total in the interaction table comes from a number of plots equal to $r \times$ (no. of levels of third factor).

(3) Also, in using [19.7] to obtain the S.S. for a two-factor interaction by subtraction, the Treatments S.S. must be replaced by the S.S. of the totals in the interior of the two-factor interaction table concerned, and the same applies to the check given in § 19.11.4.

The S.S.(ABC) is then obtained as

$$\text{Treatments S.S.} - \text{S.S.}(A) - \text{S.S.}(B) - \text{S.S.}(C) - \text{S.S.}(AB) - \text{S.S.}(AC) - \text{S.S.}(BC) \qquad [19.18]$$

although, if two of the factors are at two levels, a direct calculation similar to that of § 19.11.2 or § 19.11.3 is possible.

19.19.2 Tests of significance may be performed in the usual way of the M.S.'s for main effects, first order interactions, and the second order interaction. A conformable partitioning of the latter, computed by a simple extension of the methods of § 19.14, may be desirable but is less likely to be required (because of the probable non-existence of any significant components), and is less likely to be able to make a meaningful contribution to the interpretation of the results (cf. § 19.18.4). For example, if A, B, and C are quantitative factors, $A'B'C''$ measures the quadratic effect of the change in $A'B'$ over the levels of C, and it becomes a trifle difficult to comprehend the implications of a treatment effect of this sort.

*19.19.3 If one of the factors, say A, is assumed to have random effects, then it can be seen from arguments similar to those of § 19.12.7 that the main effects of B and C and the interaction BC should be tested against the M.S.'s for AB, AC, and ABC, respectively.

19.19.4 The significance of the triple interaction complicates the presentation and interpretation of results. A three-factor interaction table of means in standard units will have to be presented in one of the three possible ways (AB over the different levels of C, AC over the different levels of B, or BC over the different levels of A); the choice of arrangement may present itself naturally (cf. § 19.14.10) or an arbitrary choice has to be made. In any case, marginal means have to be presented in such a way that the three first order interaction tables can also be scrutinized. If the triple interaction is non-significant a presentation of the first order interaction tables is sufficient, and much simpler to follow (cf. § 22.8.1).

Example 19.3 Two varieties of maize (A and B) were grown on plots either infested artificially with witchweed (I) or uninfested (U), and in combination with four fertilizer-manurial treatments:

O = control
P = 400 lb. superphosphate per morgen
PM = P + 10 tons farmyard manure per morgen
PNK = P + 150 lb. sulphate of ammonia per morgen + 100 lb. Muriate of potash per morgen

There were two replications in randomized blocks, and the plan and yields of grain in lb. per $\frac{1}{200}$ morgen plot are shown below. Analyse the data, presenting the results in bags per morgen, given that 1 bag = 200 lb. (Data from Saunders and Rayner, *Statistical methods with special reference to field experiments*)

Block 1

BIPM 13·5	*AIO* 10·4	*BIP* 11·8	*BUPM* 22·3
BUO 12·8	*BUPNK* 17·1	*BUP* 16·9	*AUPM* 24·9
AIPM 15·8	*AIP* 12·5	*BIO* 9·5	*AUPNK* 19·9
BIPNK 11·6	*AUO* 14·8	*AIPNK* 11·3	*AUP* 19·7

Block 2

BUP 16·0	*AUP* 18·0	*BIPM* 13·4	*AIP* 10·1
AIO 10·0	*BUO* 13·0	*AIPNK* 11·4	*BIPNK* 9·2
BIP 9·5	*BIO* 9·6	*BUPNK* 16·6	*AUO* 14·0
AUPNK 19·2	*AUPM* 22·0	*BUPM* 20·0	*AIPM* 13·6

Computation sheet

Notation

V = Varieties
I = Infestation
F = Fertilizers

Skeleton analysis of variance

	D.F.		D.F.
Blocks	1	V	1
		I	1
		F	3
Treatments	15....	VI	1
		VF	3
Error	15	IF	3 (A)
Total	31	VIF	3
		Treatments	15

(B) Treatment	Block 1	Block 2	Treatment totals
AIO	10·4	10·0	20·4
AIP	12·5	10·1	22·6
AIPM	15·8	13·6	29·4
AIPNK	11·3	11·4	22·7
AUO	14·8	14·0	28·8
AUP	19·7	18·0	37·7
AUPM	24·9	22·0	46·9
AUPNK	19·9	19·2	39·1
BIO	9·5	9·6	19·1
BIP	11·8	9·5	21·3
BIPM	13·5	13·4	26·9
BIPNK	11·6	9·2	20·8
BUO	12·8	13·0	25·8
BUP	16·9	16·0	32·9
BUPM	22·3	20·0	42·3
BUPNK	17·1	16·6	33·7
Block totals	244·8	225·6	470·4

C.F. = 6,914·88

Total S.S. = 7,487·08
= 6,914·88
572·20

Blocks S.S. = 11·52 (C)

Treatments S.S. = 7,466·75
6,914·88
551·87

Preliminary analysis of variance **(D)**

Source	D.F.	S.S.	M.S.
Blocks	1	11·52	
Treatments	15	551·87	
Error	15	8·81	0·587
Total	31	572·20	

Varieties × Infestation: **(E)**

	I	*U*	Totals
A	95·1	152·5	247·6
B	88·1	134·7	222·8
Totals	183·2	287·2	470·4 **(F)**

No. of plots per single entry = 8 **(G)**

	Linear **(H)** function	S.S.
V	24·8	19·22
I	104·0	338·00
VI	−10·8	3·64
		360·86

(I) S.S. body of table = 7,275·74
6,914·88
360·86

Varieties × Fertilizers:

	O	*P*	*PM*	*PNK*	Totals
A	49·2 **(J)**	60·3	76·3	61·8	247·6 **(K)**
B	44·9	54·2	69·2	54·5	222·8
A + *B*	94·1	114·5	145·4	116·3	470·4
A − *B*	4·3	6·1	7·1	7·3	24·8

No. of plots per single entry = 4

S.S.(F) = 7,082·62 **(L)**
6,914·88
167·74

S.S.(VF) = 19·93 **(M)**
19·22
0·71

S.S. body of table = 7,102·55
6,914·88
187·67 **(N)**

Infestation × Fertilizers: (O)

	O	P	PM	PNK	Totals
I	39·5	43·9	56·3	43·5	183·2
U	54·6	70·6	89·2	72·8	287·2
$I+U$	94·1	114·5	145·5	115·3	470·4
$U-I$	15·1	26·7	32·9	29·3	104·0

No. of plots per single entry = 4

S.S.(IF) = 360·22 − 338·00 = 22·22

S.S. body of table = 7,442·85 − 6,914·88 = 527·97 (P)

Analysis of variance

Source	D.F.	S.S.	M.S.	F
Blocks	1	11·52		
Varieties (V)	1	19·22	19·22	32·7**
Infestation (I)	1	338·00	338·00	**
Fertilizers (F)	3	167·74	55·91	**
VI	1	3·64	3·64	6·2*
VF	3	0·71	0·24	—
IF	3	22·22	7·41	12·6**
VIF	3	0·34(Q)	0·11	—
Error	15	8·81	0·587	
Total	31	572·20		

Subdivision of F and IF: (R)

	O	P	PM	PNK	$\Sigma\lambda^2$
Control v. phosphate	−3	1	1	1	12
Phosphate alone v. (phosphate + supplements)	0	−2	1	1	6
M v. NK in presence of P	0	0	−1	1	2

	I (S)	U	$I+U$(T)	$U-I$(U)	Divisor
$P+PM+PNK-3$(Control)	25·2	68·8	94·0	43·6	96
$PM+PNK-2P$	12·0	20·8	32·8	8·8	48
$PNK-PM$	−12·8	−16·4	29·2	3·6	16

Auxiliary analysis of variance

Source	D.F.	S.S.	M.S.	F
Phosphate treatments	1	92·04	92·04	**
Supplements v. P alone	1	22·41	22·41	**
PM v. PNK	1	53·29	53·29	**
Fertilizers	3	167·74 (V)		
I × (Phosphate treatments)	1	19·80	19·80	**
I × (Supplements v. P alone)	1	1·61	1·61	N.S.
I × (PM v. PNK)	1	0·81	0·81	N.S.
$I \times F$:	3	22·22 (V)		
Error	15		0·587	

S.E. of a single yield = 0·766

$$\text{C.V.} = \frac{0\cdot766}{470\cdot4} \times 32 \times 100 = 5\cdot2\%.$$

S.E.'s and L.S.D.'s:

Treatment totals (2 plots) $\begin{cases} \text{S.E.} = \sqrt{(2 \times 0\cdot587)} = 1\cdot084 \\ \text{L.S.D.'s} = \sqrt{(2 \times 2 \times 0\cdot587)} \times t \text{ (15 D.F.)} \end{cases}$

$$= 1\cdot532 \times \begin{cases} 2\cdot131 \\ 2\cdot947 \end{cases}$$

$$= 3\cdot26\ (5\%)$$
$$4\cdot51\ (1\%)$$

Totals in $V \times F$ or $I \times F$ table (4 plots) $\begin{cases} \text{S.E.} = 1 \cdot 532 \\ \text{L.S.D.'s} = \sqrt{(8 \times 0 \cdot 587)} \times t \text{ (15 D.F.)} \end{cases}$

$$= 2 \cdot 167 \times t$$
$$= 4 \cdot 62 \text{ (5\%)}$$
$$6 \cdot 39 \text{ (1\%)}$$

Totals in $V \times I$ table or F totals (8 plots) $\begin{cases} \text{S.E.} = 2 \cdot 167 \\ \text{L.S.D.'s} = \sqrt{(16 \times 0 \cdot 587)} \times t \end{cases}$

$$= 3 \cdot 064 \times t$$
$$= 6 \cdot 53 \text{ (5\%)}$$
$$9 \cdot 03 \text{ (1\%)}$$

V or I totals (16 plots) $\begin{cases} \text{S.E.} = 3 \cdot 064 \\ \text{L.S.D.'s} = \sqrt{(32 \times 0 \cdot 587)} \times t \end{cases}$

$$= 4 \cdot 335 \times t$$
$$= 9 \cdot 24 \text{ (5\%)}$$
$$12 \cdot 78 \text{ (1\%)}.$$

Conversion factors:

(1) Treatment totals to means in bags per morg. $= \frac{200}{2} \times \frac{1}{200} = 0 \cdot 5$
(2) Totals of 4 plots to means in bags per morg. $= 0 \cdot 25$
(3) Totals of 8 plots to means in bags per morg. $= 0 \cdot 125$
(4) Totals of 16 plots to means in bags per morg. $= 0 \cdot 0625$

Presentation of results

TREATMENT MEANS IN BAGS PER MORG. (W)

		O	*P*	*PM*	*PNK*	Mean
Variety *A*	Infested	10·20	11·30	14·70	11·35	11·89
	Uninfested	14·40	18·85	23·45	19·55	19·06
	Mean	12·30	15·08	19·08	15·45	15·48
Variety *B*	Infested	9·55	10·65	13·45	10·40	11·01
	Uninfested	12·90	16·45	21·15	16·85	16·84
	Mean	11·22	13·55	17·30	13·62	13·93
Mean of Varieties *A* and *B*	Infested	9·88	10·98	14·08	10·88	11·45
	Uninfested	13·65	17·65	22·30	18·20	17·95
	Mean	11·76	14·31	18·18	14·54	14·70

S.E.'s and L.S.D.'s:

	S.E.	L.S.D. (5%)	L.S.D. (1%)
Single treatment mean	0·542	1·63	2·26
Mean of 2 entries	0·383	1·16	1·60
Mean of 4 entries	0·271	0·82	1·13
Mean of 8 entries	0·192	0·58	0·80

C.V. = 5·2%

Conclusion. (X) Although Variety *A* has given a yield of 1·55 bags per morg. better than Variety *B on the average* (this difference being very highly significant), under infested conditions its superiority is only 0·88 bags per morg., (Y) and, although this difference is significant at the 5% level, it does not appear that the use of Variety *A* in preference to Variety *B* will make a very marked contribution towards offsetting the effects of witchweed. The difference remains practically constant over the fertilizer-manurial treatments. (Z)

Averaged over both varieties and over uninfested and infested plots the phosphate treatments have all given a marked increase in yield over the control; the response to phosphate alone is the highly significant one of 2·55 bags per morg., and although the additional response to *NK* is negligible, that to manure is 3·85 bags per morg., which is very highly significant. Unfortunately, however, under infested conditions the effect of phosphate is much less (AA), the response to phosphate alone being reduced to 1·10 bags per morg., which is not quite significant. The response to *NK* in the presence of *P* is again negligible, (AB) but the response to manure is fairly well maintained (AB) at 3.10 bags per morg., which is very highly significant.

The experiment has not been very successful in indicating a counter to witchweed, since the best of the treatments under infestation (Variety *A* + phosphate + manure with a yield of 14·70 bags per morg.) is very much inferior to the corresponding treatment under uninfested conditions. It is, however, as good as the control plots under uninfested conditions.

Notes on the computations

It is to be noted that the fertilizer-manurial treatments have been treated as a single, factor (which we call for short "fertilizers") in the designing of the experiment, which is therefore a $2 \times 2 \times 4$ factorial arrangement with 16 treatment combinations. If manure for example, were a separate factor, it would appear in combination with *NK* as well as with no phosphate, but these combinations are excluded in the present arrangement.

(A) Obtain the D.F. for all interactions by the product rule, and check by addition to the D.F. for treatments.

(B) The usual block-treatment table is drawn up, and the G.T. checked down and across. It is helpful to place the treatments in a systematic order.

(C) Short-cut: $\dfrac{(248\cdot8 - 225\cdot6)^2}{32}$ — Formula [10.10].

(D) Optional.

(E) For computation of main effects and two-factor interactions, the three two-factor interaction tables are drawn up in turn. In the $V \times I$ table, for example,

$$95\cdot1 = \text{Total of all plots with combination } AI$$
$$= 20\cdot4 + 22\cdot6 + 29\cdot4 + 22\cdot7,$$

and so on.

(F) Check down and across to G.T.

(G) This gives the divisor for the total S.S. of the interaction table, commonly called the S.S. for the "body of the table", and is helpful in determining the other divisors required.

(H) The general method here is to use [19.6] and [19.7], but since this is a 2×2 table we proceed in the manner explained in § 19.11.3.

$$24\cdot8 = 247\cdot6 - 222\cdot8$$
$$104\cdot0 = 287\cdot2 - 183\cdot2$$
$$-10\cdot8 = 95\cdot1 + 134\cdot7 - 152\cdot5 - 88\cdot1$$

The divisor is 32 (= total number of plots) for all three S.S.'s.

(I) $\frac{1}{8}(95\cdot1^2 + 152\cdot5^2 + 88\cdot1^2 + 134\cdot7^2) - C.F.$ This must check to S.S.(V) + S.S.(I) + S.S.(VI).

(J) $49\cdot2$ = Total of all plots with combination $AO = 20\cdot4 + 28\cdot8$.

(K) Checks here: A and B totals must be as in previous table. Marginal totals check down and across to G.T. Total of differences $A - B = 247\cdot6 - 222\cdot8$ = linear function for V previously obtained. The procedure for the S.S.'s is as explained in § 19.11.2.

(L) Divisor is 8 (= 2×4 or $32 \div 4$), c.f. Example 19.1, Note I. Only one main effect S.S. need be computed, because the other has already been obtained.

(M) Formula [19.8]. Divisor is same as for the fertilizer totals (see previous note). C.F. = S.S.(V).

(N) This must check to S.S.(V) + S.S.(F) + S.S.(VF).

(O) The computations here are identical with those for the $V \times F$ table, and Notes I to M should be consulted where needed. The marginal totals must check with those previously obtained.

(P) $527\cdot97$ = S.S.(I) + S.S.(F) + S.S.(IF). Check.

(Q) The standard method is to apply [19.18], but actually here the S.S.(VIF) can be evaluated directly since two of the factors are at 2 levels each (see § 19.19.1). To do this, calculate VI as $AI + BU - AU - BI$ for each fertilizer level separately. These values are as follows: $-1\cdot7$, $-3\cdot5$, $-2\cdot1$, $-3\cdot5$, totalling to $-10\cdot8$ = linear function for VI, (e.g. $-1\cdot7 = 20\cdot4 + 25\cdot8 - 28\cdot8 - 19\cdot1$). The S.S.($VIF$) is then $\frac{1}{8}(1\cdot7^2 + 3\cdot5^2 + 2\cdot1^2 + 3\cdot5^2) - C.F. = 3\cdot98 - 3\cdot64 = 0\cdot34$, the C.F. being the S.S.(VI).

(R) The levels of factor F comprise a classified set. Subdivisions of the S.S.(F) and the S.S.(IF) are therefore appropriate. No single D.F. of VF or VIF could be significant here, or else the subdivision of these S.S.'s would also need to be considered. The proposed subdivision is orthogonal.

(S) To obtain the figures in these columns, the coefficients are applied to the totals in the I and U rows (separately), of the $I \times F$ interaction table.

(T) The $I + U$ totals give the result of applying the coefficients to the fertilizer totals, and from this column we obtain the S.S.'s for the main effect comparisons.

(U) The S.S. for $I \times$ (Control *v.* phosphate) could be obtained as $\frac{1}{48}(25\cdot2^2 + 68\cdot8^2) - \frac{1}{96}(94\cdot0^2)$, and this method would be followed in the general case when I has more than 2 levels, but here it is easier to form the difference. The same applies to the other two linear functions. The divisors are $2 \times 4 \times \Sigma\lambda^2$.

If it had been necessary to subdivide *VIF*, it would have been necessary to evaluate each of these 3 linear functions for each combination of V and I levels, thus obtaining a $V \times I$ interaction table for each of the 3 linear functions. The S.S. for the interaction in each of these tables (with divisor $r\Sigma\lambda^2$) would give the orthogonal component S.S.'s of *VIF*.

The linear functions in both the "$I + U$" and "$U - I$" columns could be tested directly by the t-test, but it is advisable to use the check given by the orthogonality.

(V) Checks to S.S. previously obtained.

(W) Although the results here can be explained in terms of main effects and first order interactions only, the full second order interaction table has been presented to illustrate the method. It will be seen that all the two-factor tables are included; the $V \times F$ table is given by the means for Variety A and the means for Variety B, the $V \times I$ table is given by the means in the final column, and the $I \times F$ table is given at the bottom.

This experiment was carried out to see if fertilizer-manure treatments could be an effective counter to the witchweed parasite of the roots of maize plants. There is therefore practically no interest in the main effect of the infestation factor, but very great interest in the $I \times F$ interaction. The table has therefore been presented so that the $I \times F$ table is shown separately for each variety.

(X) The type of conclusion drawn here is atypical because of the purpose of the experiment (see previous note); for example, it would be absurd to say that "No infestation is a better treatment than infestation with witchweed".

(Y) The significance of *VI* points to this.

(Z) *VIF* non-significant.

(AA) The significance of ($I \times$ Phosphate treatments) points to this.

(AB) Shown by the non-significance of the remaining components of *IF*. Only the first of the three components of F varies in infested and uninfested conditions.

19.20 Experiments with more than three factors

19.20.1 The necessary extension of the theory and method of analysis to cover four or more factors should be fairly obvious. With 4 factors (A, B, C, D) the partitioning of the Treatments S.S. will be into main effects A, B, C, D, first order interactions AB, AC, AD, BC, BD, CD, second order interactions ABC, ABD, ACD, BCD, and the **third order or four-factor interaction** $ABCD$. The latter represents variation in the second order interactions over the levels of the other factor, e.g. variation in ABC over the levels of D, in ABD over the levels of C, etc. The D.F. of all components are given by the product rule.

19.20.2 With many factors it is quite easy to omit an interaction inadvertently from the full list. It is helpful here to know that for a given number of factors the numbers of interactions of different orders are given by the binomial coefficients (Table 7.3). These are reproduced here in Table 19.7, which may be extended if required with the aid of Pascal's triangle.

19.21 High order interactions as an estimate of error

19.21.1 The existence of a third order interaction indicates that the effects of the four factors in combination cannot be described without taking into account the specific joint action which occurs only when these particular four factors are combined. In other words, the effects cannot be described in terms

Table 19.7 No. of interactions of different orders with varying numbers of factors

	Number of factors			
	2	3	4	5
Main effects	2	3	4	5
1st order	1	3	6	10
2nd order		1	4	10
3rd order			1	5
4th order				1

of main effects alone, nor in terms of main effects modified by the interactions of pairs of factors, even when allowance is made for the possibility that the latter may vary over the levels of one of the other factors. The chances that a genuine effect of this nature will exist are rather remote, and any interpretation of such an effect would be difficult. As a result, the M.S.'s corresponding to third and higher order interactions are usually expected (at least in agricultural experiments) to be of the order of experimental error, and it is common for the occasional significance of a high order interaction to be dismissed as the odd $\frac{1}{20}$ or $\frac{1}{100}$ chance, although this should not be done automatically.

19.21.2 Advantage is taken of this by making use of such high order interactions to estimate error. The purpose here is twofold: (i) to save computation; (ii) to increase the D.F. available for estimation of error. Only S.S.'s for main effects and interactions of low order are computed, those of higher order being pooled with error, i.e. included in an "error" S.S. obtained by subtraction. There is, of course, some danger of error being over-estimated, if genuine high order interactions exist. However, this possibility is not taken very seriously, and even second order interactions are sometimes used for estimation of error, especially the less easily interpretable components (cf. §§ 19.18.4 and 19.19.2).

19.21.3 With several factors the number of treatments may be so large that it would be impossible to have even two replicates. However, it is possible to use only *a single replicate*, which provides no genuine error D.F. at all, but to allot certain high order interactions (in advance, to avoid bias) for estimation of error.

19.21.4 This idea of dispensing with replication (see, however, § 19.23.2) is carried even further in designs with *fractional replication*, where only a fraction of all the possible treatment combinations is used.

19.21.5 The use of high order interactions to estimate error is equivalent to making an assumption that they are negligible and consequently including no such high order interaction terms in the model for the treatment effects.

*19.22 Dummy treatments

19.22.1 Certain factor-pairs are such that among the treatment combinations are a number of identical treatments. An example is: 3 levels of nitrogen (nil, 100 lb., 200 lb.) and 4 types of nitrogen; the four combinations with zero nitrogen are identical. Differences between such treatments can only be due to error, and they are therefore called "dummy treatments". It is really comparisons between them which are "dummy". Strictly speaking, variation due to dummy comparisons should be included in error, especially since its presence in the Treatments S.S. will water down genuine treatment effects. The idea has been encountered before (Example 18.2).

19.22.2 No methods of dealing with dummy treatments will be presented. No great harm can result from following standard methods ignoring dummies.

19.23 An appraisal of the factorial method

19.23.1 The inadequacy of the single-factor approach, which served to introduce this chapter, can now be more glaringly exposed. Actually, if an experimenter wished to ascertain simultaneously the best variety of a crop and the best phosphate dressing, he probably would not, under the single-factor system, conduct the series of 7 experiments mentioned in § 19.1.3. (If he did, he might at least become aware of an interaction between the factors if it were sufficiently pronounced.) What he would probably do would be to conduct two experiments—a varietal trial at some guessed optimal level of phosphate, and a fertilizer experiment with some selected variety.

19.23.2 Such an approach can be successful only if the factors do not interact, but, even if they do not, it is an inefficient method. Assume that there are 4 varieties and 3 levels of phosphate and that 36 plots are devoted to each of the two single-factor experiments. Then for the same number of plots we could have a 4×3 factorial experiment with 6 replications. In the single-factor experiments comparisons between varieties will be based on means of 9 plots, and comparisons between phosphate levels on means of 12 plots. In the two-factor experiment, however, the corresponding comparisons will, in the absence of interaction, be based on means of 18 and 24 plots respectively, i.e. *all the plots* contribute to the estimation of both main effects. It is thus seen that the factorial experiment has a property called "*hidden replication*" in that a design with only 6 "absolute replications" can effectively muster 18 and 24 replications for the main effect comparisons. It is true that this property vanishes in the presence of interaction (when the effects of each factor have to be considered at the separate levels of the other factor), but in these same circumstances the single-factor experiments fail completely, being unable to contribute any information about interaction.

19.23.3 Recommendations based on multifactorial experiments have *a wider inductive basis* than would otherwise be obtained. If the phosphate responses are similar for all varieties, the optimum level can be recommended with greater confidence than if the results were based on a single variety only; and if interaction is found to exist, the experimenter is prevented from making a recommendation which would not have been correct for some varieties.

19.23.4 In practice it is impossible to specify standard conditions for a wide range of potential factors such as those listed in § 19.1.1, the variation of which may have a bearing on the result of an experiment. A solution is deliberately to vary the conditions by the introduction of additional factors (though this in no way contradicts the previously asserted necessity for uniform conditions in order to ensure low experimental error). If any factor so introduced does not interact with the primary factors in the experiment, the results will have a broader basis; if there is interaction, this is an important fact which otherwise would not have been discovered. Thus, rather than conduct a factorial experiment with two replications, it would probably be better to add some factor of interest with two levels and conduct the experiment as a single replicate.

19.23.5 On the other hand, it is obviously impossible to study all possible factors in all circumstances and some cutting down may be necessary. For instance, in Example 19.3 it was pointed out how farmyard manure was introduced as a treatment, but not as a separate factor. Yet there is no doubt that many supposedly "simple" experiments are, in fact, more difficult to interpret than a fully factorial arrangement, and in many cases the absence of only a single treatment combination causes difficulties of interpretation as well as a loss of efficiency and orthogonality (cf. § 8.3.1).

19.23.6 Without any doubt, however, one of the main difficulties with factorial experiments is the rapid multiplication of treatment combinations. Even with only four factors the minimum number of treatment combinations is 16, which means that an ordinary Latin square is virtually impossible, and that the reasonable maximum number of treatments with an ordinary randomized blocks design has been reached (§ 13.13.2). There is also the problem of adequate replication without having an impossibly large experiment. The latter difficulty is really resolved by hidden replication. The problem of block size can be overcome on the lines mentioned in § 13.13.3 (incomplete blocks), aided by a device of design which will be introduced in Chapter 22. The Latin square can also be adapted to some extent by devices of the same sort, but the greater efficiency of the Latin square has to be largely forgone. From the difficulty of sheer numbers of treatments (even without replication) there is some relief to be found in fractional replication (§ 19.21.4). It is to be noted, however, that the factorial principle is of universal application in experimentation, and there are undoubtedly fields where such matters as large numbers of treatments and block size are relatively unimportant.

19.23.7 The numbers of levels of the factors, of course, contribute as much to the number of treatment combinations as the number of factors, and some cutting down is frequently necessary here also. Fortunately, the number of levels and the number of factors are to some extent complementary and it is possible to recognize two extreme situations:

(1) *Exploratory experiments* with a large number of factors, all thought to have a bearing on the result, but often at only two levels each. Experiments of this nature with levels of the "nil and plenty" type enable irrelevant factors to be discarded with the minimum amount of wasted effort.

(2) Experiments with few factors where it is necessary to make a careful determination of optimal levels. Here a factor may have several levels.

In general the principles set out in §§ 17.17.20 and 17.17.21 may be followed in respect of quantitative factors, but with an eye on the need for compromise if the number of treatment combinations is too large.

19.23.8 On their introduction factorial experiments did not entirely escape criticisms, but in the light of 30 years' experience (longer at Rothamsted), these can be seen in their true perspective. They are:

(1) Complexity of lay-out. In fact, it is no more difficult to follow a plan with factorial treatments than any other list of treatments, provided care is taken.

(2) Some "silly" treatment combinations occur in factorial designs. The answer here is that they may not be as silly as they seem, while considerations such as those mentioned in § 19.23.5 are also very relevant.

(3) Comparisons between individual pairs of treatment means are very imprecise owing to the low number (usually) of absolute replications. This is correct, since the results of factorial experiments are based on group comparisons (§ 11.6) unless interactions of the highest order are present. But it is not really the purpose of factorial experiments to make such individual comparisons, and the experimenter should choose his treatments accordingly.

In fact, the factorial concept is without doubt one of Fisher's most notable contributions to the logic of experimentation, and as so often when such an "obvious" step is made, the wonder is that it was not thought of before. Wisdom after the event is, however, a relatively cheap commodity.

EXERCISES

19.1 In a fodder-grass experiment, 4 cutting rotations were tested with 3 varieties of grass. The design was in randomized blocks and the cutting rotations are as follows:

A = cropped every 45 days (8 times per year)
B = „ „ 90 „ (4 „ „ „)
C = „ „ 120 „ (3 „ „ „)
D = „ „ 180 „ (2 „ „ „)

Analyse the following yields from the first year of the experiment which are in lb. per $\frac{1}{57}$ acre plot:
(From *Statistical technique in agriculture research* by D. D. Paterson, Copyright, 1939 by, the McGraw-Hill Book Company Inc. Used by permission of McGraw-Hill Book Company.)

Blocks	Elephant Grass				Guatemala Grass				Uba Cane			
	A	*B*	*C*	*D*	*A*	*B*	*C*	*D*	*A*	*B*	*C*	*D*
1	96	187	222	109	146	252	246	277	115	298	220	430
2	70	163	125	97	133	181	263	293	143	220	341	371
3	77	143	134	133	154	224	194	260	117	234	258	484
4	80	179	173	113	146	248	190	325	120	253	297	460

19.2 Below are the plan and green weights (in lb. per plot of 20 ft. × 10 ft.) observed in a trial with kale. Analyse, presenting results in tons per acre (1 ton = 2240 lb.). Treatment symbols:

D = 15 tons dung per acre
O = No dung
0 = Sulphate of ammonia—Nil
1 = „ „ „ 0·2 cwt. N per acre
2 = „ „ „ 0·4 cwt. N per acre
3 = „ „ „ 0·8 cwt. N per acre

*D*2 230	*O*3 258	*O*2 242	*D*0 194	*O*0 145	*D*1 268	*D*3 285	*O*1 185
*D*0 167	*D*1 176	*D*3 314	*O*2 196	*D*2 230	*O*1 198	*O*3 231	*O*0 136
*O*3 240	*O*2 189	*D*2 280	*O*0 127	*D*1 240	*D*0 212	*O*1 148	*D*3 298
*D*1 155	*D*0 142	*O*1 194	*D*2 222	*O*3 255	*D*3 318	*O*0 118	*O*2 208
*O*2 180	*D*2 212	*D*0 204	*D*3 285	*O*1 171	*O*0 152	*D*1 258	*O*3 251
*O*0 124	*O*1 176	*O*3 247	*D*1 155	*D*3 283	*O*2 170	*D*2 254	*D*0 208
*O*1 177	*D*3 306	*O*0 146	*O*3 249	*D*0 203	*D*2 260	*O*2 184	*D*1 206
*D*3 275	*O*0 145	*D*1 289	*O*1 213	*O*2 248	*O*3 274	*D*0 249	*D*2 258

(From Rothamsted Report, 1932, Page 167)

19.3 In an experiment at Cedara to study the control of late blight on potatoes, the field plan and plot yields were as follows:

Block 1	v_2f_1 34·1	v_2f_3 41·7	v_2f_2 36·0	v_1f_2 55·9	v_3f_1 35·9
	v_3f_4 76·3	v_2f_4 63·9	v_2f_0 21·3	v_3f_0 36·6	v_3f_2 54·2
	v_3f_3 72·2	v_1f_4 100·9	v_1f_1 55·8	v_1f_3 72·7	v_1f_0 39·0
Block 2	v_1f_1 51·9	v_1f_4 88·3	v_3f_0 35·7	v_3f_4 88·2	v_2f_1 21·0
	v_3f_2 42·9	v_1f_3 64·7	v_1f_0 42·2	v_2f_2 24·5	v_2f_3 41·0
	v_2f_4 62·1	v_3f_1 33·1	v_1f_2 65·6	v_3f_3 66·5	v_2f_0 11·4
Block 3	v_3f_4 66·3	v_2f_2 26·4	v_1f_2 60·1	v_3f_3 52·7	v_3f_0 35·9
	v_3f_2 39·1	v_2f_3 59·8	v_1f_0 45·2	v_2f_0 11·8	v_2f_4 69·0
	v_1f_3 75·1	v_2f_1 28·3	v_1f_1 47·2	v_3f_1 53·1	v_1f_4 102·1

The treatments were

Varieties (V)	Fungicides (F)
v_1 = B.P. 1	f_0 = control
v_2 = Up-to-date	f_1 = sprayed 4 times with Copper oxychloride
v_3 = Pimpernel	f_2 = sprayed 10 times with Copper oxychloride
	f_3 = sprayed 4 times with Dithane M-22
	f_4 = sprayed 10 times with Dithane M-22

Analyse the data presenting results in bags per morgen, given that the net plot consisted of 4 data rows 16½ ft. long with 3 ft. spacing between rows, and that

1 bag = 150 lb.
1 morgen = 10,244 sq. yards.

(Data from Department of Plant Pathology and Microbiology, Natal Region, Department of Agricultural Technical Services)

19.4 The effect of various media on the oxygen uptake of poultry spermatazoa was tested in an experiment in which the factors were as follows:

Buffers	Diluents	Nutrients
b_0 = Unbuffered	d_0 = Sodium chloride	n_0 = nil
b_1 = Phosphate	d_1 = Mixture of salts	n_1 = Glucose
b_2 = Glutamate		

There were 5 "runs" (replications), a different source of semen being used for each run. The following data represent oxygen uptake in microlitres per milligram of dry tissue per hour. Analyse the results.

Medium	Run 1	2	3	4	5
$b_0d_0n_0$	5·70	9·87	14·27	9·23	18·14
$b_0d_0n_1$	12·75	11·83	15·07	13·03	20·87
$b_0d_1n_0$	8·85	13·57	11·02	9·83	13·42
$b_0d_1n_1$	20·53	13·00	12·75	11·86	15·37
$b_1d_0n_0$	14·04	12·71	12·74	9·04	12·92
$b_1d_0n_1$	10·40	12·23	17·56	13·10	22·37
$b_1d_1n_0$	13·44	10·66	11·50	8·26	15·72
$b_1d_1n_1$	9·33	10·73	17·27	8·97	23·25
$b_2d_0n_0$	10·36	6·07	14·15	4·54	20·01
$b_2d_0n_1$	10·72	11·73	21·32	10·08	27·39
$b_2d_1n_0$	15·94	12·96	16·81	8·29	24·00
$b_2d_1n_1$	11·16	13·84	18·41	13·88	23·29

(Data from B. R. Tulloch, Department of Biochemistry, University of Natal)

19.5 The data from an experiment to test the effects of different fertilizers on potatoes are set out beneath. The treatments consisted of all combinations of the following factors:

Sulphate of Ammonia (N)		Sulphate of Potash (K)		Superphosphate (P)
N_0 = nil	×	K_0 = nil	×	P_0 = nil
N_1 = 0·3 cwt. N per acre		K_1 = 0·5 cwt. K_2O per acre		P_1 = 0·5 cwt. P_2O_5 per acre
N_2 = 0·6 cwt. N per acre		K_2 = 1·0 cwt. K_2O per acre		

Treatment	Block 1	2	3	4
$N_0K_0P_0$	254	272	270	292
$N_0K_0P_1$	235	283	267	293
$N_0K_1P_0$	276	267	248	259
$N_0K_1P_1$	259	293	266	262
$N_0K_2P_0$	244	266	256	280
$N_0K_2P_1$	247	265	268	272
$N_1K_0P_0$	241	279	272	310
$N_1K_0P_1$	239	279	266	291
$N_1K_1P_0$	270	307	273	354
$N_1K_1P_1$	286	326	306	299
$N_1K_2P_0$	254	291	281	292
$N_1K_2P_1$	270	289	324	281
$N_2K_0P_0$	260	286	290	291
$N_2K_0P_1$	284	275	281	279
$N_2K_1P_0$	290	298	280	293
$N_2K_1P_1$	301	307	288	289
$N_2K_2P_0$	284	269	297	310
$N_2K_2P_1$	296	295	291	314

Analyse the data, presenting the results in bags per acre. (Yields are given in lb.; plot size = $\frac{1}{100}$ acre; 1 bag = 150 lb.)